Mathematics 106 — Calculus II

Exam 1

February 6, 2009

Your Name: SOLUTION GUIDE

There are 6 problems in this exam. On each problem, you must show all your work, or otherwise thoroughly explain your conclusions. **There is always at least one step preceding a final answer.** *Units may be requested for your final answer; a point deduction will apply if they are omitted.*

On the portion of the exam marked No Calculator*, you will be allowed 20 minutes during which your calculator must be closed and put away. If you finish this section early, you may hand in your work early. However,* **only after you hand in the "no calculators" section will you be permitted to use a calculator.**

You will have 55 minutes to complete this exam.

Question	Point Value	Your Score
No Calc.	50	
1	30	
2	35	
3	35	
Total	150	

NO CALCULATOR PORTION

Name: ______________________

Math 106-C (Salomone)

Exam 1

Score (50 possible):

Show all your work!

Problem 1-NC. (20 points) Use the Fundamental Theorem of Calculus to evaluate the following integrals:

(a) (10 points) $\displaystyle\int \frac{1}{\sqrt{x}\left(5+\sqrt{x}\right)^2}\,dx$

Use the substitution $u = 5 + \sqrt{x}$. This is perfectly set up, since $du = \frac{1}{2\sqrt{x}}dx$ also appears in the integral.

$$\begin{aligned}\int \frac{1}{\sqrt{x}\left(5+\sqrt{x}\right)^2}\,dx &= \int \frac{2}{u^2}\,du \\ &= -\frac{2}{u} + C \\ &= \boxed{-\frac{2}{5+\sqrt{x}} + C.}\end{aligned}$$

(b) (10 points) $\displaystyle\int_0^1 e^{e^x+x}\,dx$

A classic case of ABC — algebra before calculus. Simplify the exponential of a sum into the product of exponentials:

$$e^{e^x+x} = e^{e^x}e^x$$

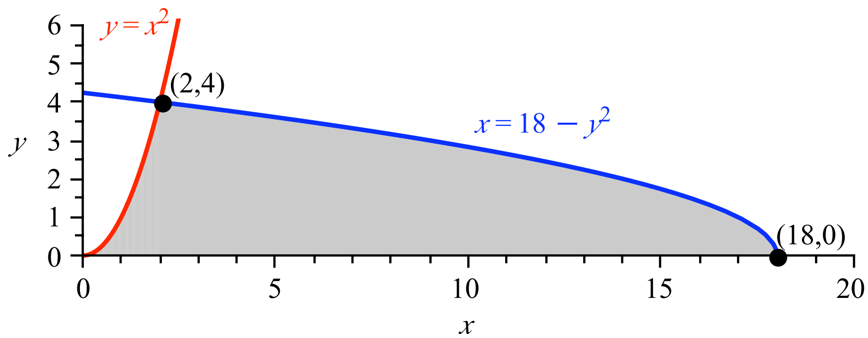
and you'll see we have another perfect setup:

$$\int_0^1 e^{e^x}\,e^x\,dx$$

Taking $u = e^x$ and $du = e^x\,dx$ gives us

$$\begin{aligned}\int_{x=0}^{x=1} e^u\,du &= \int_{u=1}^{u=e} e^u\,du \\ &= e^u\Big|_1^e \\ &= \boxed{e^e - e.}\end{aligned}$$

Questions 2 — 3 refer to the graph shown below.



Problem 2-NC. (20 points) Compute the *exact area* of the shaded region.

This can be done with either vertical or horizontal strips, but horizontal is easier. (With vertical, you would have to split the region in two.)

Therefore, we should solve each equation for x to determine the identities of the "left" and "right" functions.

$$y = x^2 \qquad\qquad x = 18 - y^2$$
$$x = \sqrt{y} \qquad\qquad x = 18 - y^2$$

The total area is then given by the integral of their difference, beginning at the x-axis with $y = 0$ and ending at their intersection point with $y = 4$:

$$\begin{aligned}\int_0^4 18 - y^2 - \sqrt{y}\,dy &= 18y - \frac{1}{3}y^3 - \frac{2}{3}y^{3/2}\Big|_0^4 \\ &= 18(4) - \frac{1}{3}(64) - \frac{2}{3}(8) \\ &= 72 - \frac{80}{3} = \boxed{\frac{136}{3}}.\end{aligned}$$

Problem 3-NC. (10 points) SET UP — BUT DO NOT EVALUATE — an integral which computes the length of the curve shown above, whose endpoints are $(2, 4)$ and $(18, 0)$.

We can either perform this integral with respect to x or with respect to y; however, the former would require us to solve the equation $x = 18 - y^2$ for x. Instead, let's compute the arclength as

$$\int_{y=0}^{y=4} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy = \int_0^4 \sqrt{1 + (-2y)^2}\,dy = \boxed{\int_0^4 \sqrt{1 + 4y^2}\,dy.}$$

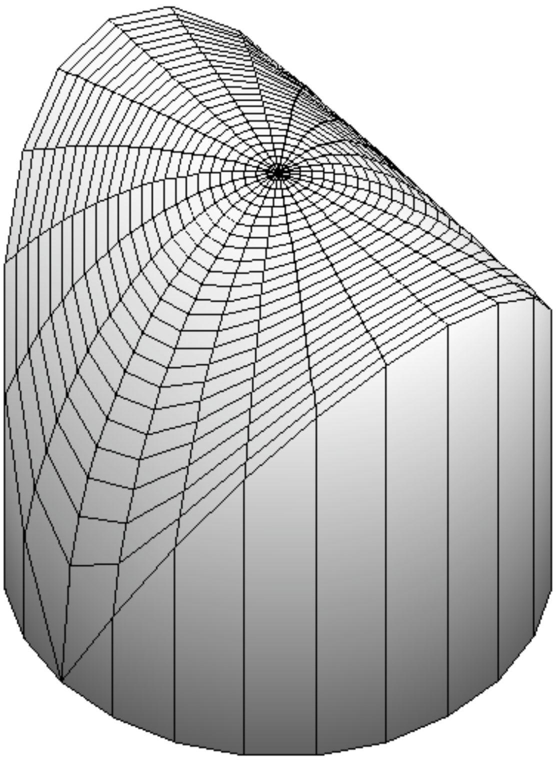
If we had set this up with respect to x, we would solve for y to obtain

$$y = \sqrt{18 - x}$$

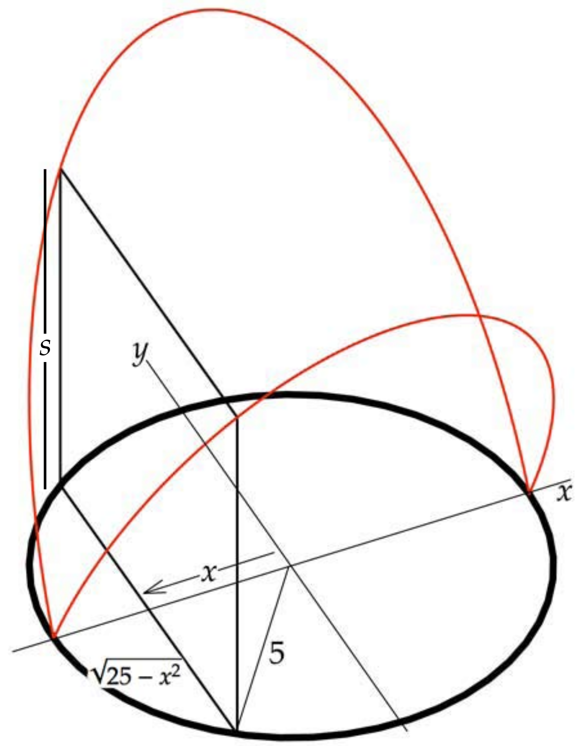
and the integral

$$\int_{x=2}^{x=18} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \boxed{\int_2^{18} \sqrt{1 + \frac{1}{4(18 - x)}}\,dx.}$$

Problem 1. (30 points) You design a camping tent whose footprint is a 10-foot-diameter circle and whose carbon-fiber frame is designed of rigid squares.



The tent...



...and a view of its *square* cross sections.

Calculate the *exact volume* this tent encloses, in cubic feet.

Each cross section at a given x value is a square. If we label the sides of this square s, then according to the diagram and the Pythagorean theorem,

$$s = 2\sqrt{25 - x^2} \qquad \text{so} \qquad s^2 = 4(25 - x^2) = 100 - 4x^2.$$

Thus the area of the cross-section at a marked x value is

$$A(x) = x^2 = 100 - 4x^2.$$

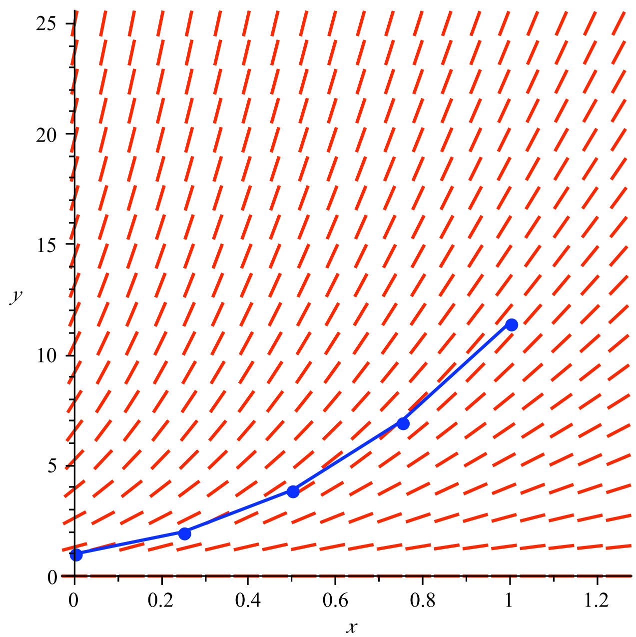
The leftmost slice occurs at $x = -5$, while the rightmost occurs at $x = 5$.

The total volume is then

$$\begin{aligned} \int_{-5}^{5} 100 - 4x^2\,dx &= 100x - \frac{4}{3}x^3\Big|_{-5}^{5} \\ &= 500 - \frac{4}{3}(125) - \left(-500 + \frac{4}{3}(125)\right) \\ &= 1000 - \frac{1000}{3} = \boxed{\frac{2000}{3}\text{ ft}^3.} \end{aligned}$$

Problem 2. (35 points) Consider the initial-value problem

$$\frac{dy}{dx} = \frac{4y}{1+x^2} \qquad y(0) = 1.$$



(a) (15 points) Use Euler's method with step size $\Delta x = 0.25$ to approximate $y(1)$, and indicate what you've done on the slope field at left. Be sure to circle your final answer.

We will need four steps of $\Delta x = 0.25$ to get from $x = 0$ to $x = 1$. Using the convenient table:

x	y	y'
0	1	4
	$+4(0.25) =$	
0.25	2	7.53
	$+7.53(0.25) =$	
0.5	3.88	12.424
	$+12.424(0.25) =$	
0.75	6.986	17.884
	$+17.884(0.25) =$	
1	$\boxed{11.457}$	—

(b) (20 points) Use separation of variables to find the exact function which solves this IVP. How well does your answer to part (a) compare with the exact value of $y(1)$?

We'll need all the x on one side of the equation, and all the y on the other:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{4y}{1+x^2} \\
\int \frac{1}{y}\,dy &= \int \frac{4}{1+x^2}\,dx \\
\ln y &= 4\arctan x + C \\
y &= e^{4\arctan x + C} = Ae^{4\arctan x}.
\end{aligned}$$

To satisfy $y(0) = 1$, remember that $\arctan 0 = 0$ (a line of slope 0 will make an angle of 0 radians with the positive x-axis). Then

$$\begin{aligned}
y(0) = Ae^{4\arctan 0} &= 1 \\
Ae^0 &= 1 \\
A &= 1
\end{aligned}$$

Thus $\boxed{y(x) = e^{4\arctan x}}$ and $y(1) = e^{4\arctan 1} = e^{\pi} \approx 23.14$. This represents an error of $|23.14 - 11.457| = 11.67$ from our estimated value.

Problem 3. (35 points) As caddymaster of Vice City Country Club, you have just hired a caddy named Tommy whom you suspect of taking golf carts for late-night joyrides around the city.

Accordingly, you install a monitor on all your golf carts which secretly records the cart's speed s at 15-minute intervals during the night. You also install a governor on each engine which limits the cart to a speed of no more than 18 mph, and acceleration of no more than 24 mph/hr.

Sunday morning, the readout from one of the carts looks suspicious:

t (hr)	21	21.25	21.5	21.75	22	22.25	22.5	22.75	23
s (mph)	1.5	5.7	10.1	15.5	17.6	14.1	10.5	4.5	0

(a) (20 points) Use a right-hand sum R_8 to estimate how far Tommy drove the golf cart during his two-hour joyride.

Luckily for us, $\Delta t = \frac{23-21}{8} = 0.25$ and our table shows exactly intervals of this size. By taking the right-hand endpoint of each interval, we get the approximation

$$\begin{aligned} R_8 &= 0.25\big(5.7 + 10.1 + 15.5 + 17.6 + 14.1 + 10.5 + 4.5 + 0\big) \\ &= 0.25(78) = \boxed{19.5 \text{ miles.}} \end{aligned}$$

(b) (15 points) If a golf cart is driven more than 25 miles in a day, the insurance company requires it be reclassified as a passenger car, and will charge higher rates. Could you plausibly accuse Tommy of having taken a 25-mile joyride, based on your answer to part (a)?

Hint: how far off can R_8 be from the exact distance he drove?

We have used a right-hand sum to estimate $\int_{21}^{23} s(t)\,dt$. According to the error bounds, the largest amount of error that R_N can commit in approximating an integral is

$$\frac{K(b-a)^2}{2N}$$

where K is an upper bound for $s'(t)$.

But $s'(t)$ is just the acceleration of the golf cart — and according to the problem, acceleration is limited to no more than 24 mph/hr. Thus $|s'(t)| \le 24$ and we can take $K = 24$.

Using $a = 21$, $b = 23$, and $N = 8$, we find that the largest amount of error R_8 could commit is

$$\left|I - R_8\right| \le \frac{24(23-21)^2}{2(8)} = 6.$$

Therefore, the actual distance Tommy drove must be between

$$R_8 - 6 = 13.5 \qquad \text{and} \qquad 25.5 = R_8 + 6$$

So it is possible, based on the data gathered, that Tommy drove more than 25 miles.

(Whether he actually did or not is determined by how much he leaned on the gas pedal in between the recorded data points. The point is, without more data, Tommy cannot prove his innocence.)