Math 205B&C Quiz 03 page 1 02/06/09 **Name** suggested solution **Section: 8am 1:10 pm**

1. Suppose $T : \mathbf{R}^p \to \mathbf{R}^q$ is the linear transformation defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 15x_1 + 10x_2 + 35x_3 \\ 16x_1 + 14x_2 + 24x_3 \\ 12x_1 + 10x_2 + 20x_3 \\ 10x_1 + 8x_2 + 18x_3 \end{bmatrix}$.

1a. What is the value of p? 3 … and the value of q? 4

1b. Find the image under T of $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

$$T\left(\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 30+0+35 \\ 32+0+24 \\ 24+0+20 \\ 20+0+18 \end{bmatrix} = \begin{bmatrix} 65 \\ 56 \\ 44 \\ 38 \end{bmatrix}$$

1c. Find the associated, or standard, matrix A of T.

$$A = \begin{bmatrix} 15 & 10 & 35 \\ 16 & 14 & 24 \\ 12 & 10 & 20 \\ 10 & 8 & 18 \end{bmatrix}$$

1d. Use A in a superaugmented matrix to find explicit conditions on $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$ that will guarantee that $\mathbf{b}$ is in the range of T.

$$\begin{array}{ccccccc} x_1 & x_2 & x_3 & b_1 & b_2 & b_3 & b_4 \end{array}$$
$$\left[\begin{array}{ccc|cccc} 15 & 10 & 35 & 1 & 0 & 0 & 0 \\ 16 & 14 & 24 & 0 & 1 & 0 & 0 \\ 12 & 10 & 20 & 0 & 0 & 1 & 0 \\ 10 & 8 & 18 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|cccc} 1 & 0 & 5 & 0 & 0 & -2 & 5/2 \\ 0 & 1 & -4 & 0 & 0 & 5/2 & -3 \\ 0 & 0 & 0 & 1 & 0 & 5 & -15/2 \\ 0 & 0 & 0 & 0 & 1 & -3 & 2 \end{array}\right]$$

The last two rows show $T(\vec{x}) = \vec{b}$, or $A\vec{x} = \vec{b}$, has a solution $\Longleftrightarrow \begin{cases} 0 = b_1 + 5b_3 - \frac{15}{2}b_4 \\ 0 = b_2 - 3b_3 + 2b_4 \end{cases}$

1e. Use (1d) to find a value for each of b_1 and b_2 so that $\mathbf{d} = \begin{bmatrix} b_1 \\ b_2 \\ 1 \\ 2 \end{bmatrix}$ is the image of some vector $\mathbf{x}$ in $\mathbf{R}^p$.

with $b_3 = 1$ and $b_4 = 2$ in the above two conditions,
we find $b_1 = 10$ and $b_2 = -1$

1f. Let $\mathbf{d}$ be as in the previous problem, and find at least one $\mathbf{x}$ for which $T(\mathbf{x}) = \mathbf{d}$.

by design $T(\vec{x}) = \vec{d}$ has a solution and in fact $\begin{cases} x_1 = (-2b_3 + \frac{5}{2}b_4) - 5x_3 \\ x_2 = (\frac{5}{2}b_3 - 3b_4) + 4x_3 \end{cases}$ (where x_3 is free) $\Rightarrow \begin{cases} x_1 = (-2+5) - 5x_3 \\ x_2 = (\frac{5}{2} - 6) + 4x_3 \end{cases} \Rightarrow \begin{cases} x_1 = 3 - 5x_3 \\ x_2 = -\frac{7}{2} + 4x_3 \\ x_3 = x_3 \end{cases}$

for example, if $x_3 = 0$ we get $\vec{x} = \begin{bmatrix} 3 \\ -3.5 \\ 0 \end{bmatrix}$; if $x_3 = 1$ we get $\vec{x} = \begin{bmatrix} -2 \\ +1/2 \\ 1 \end{bmatrix}$

1g. Is T *onto* $\mathbf{R}^q$? Explain in terms of the definition or any equivalent statement about A.

NO. In 1d we see that $T(\vec{x}) = \vec{b}$ does NOT always have a solution since there are CONDITIONS $\vec{b}$ must satisfy.
(or in terms of A, RREF(A) has rows of 0's, $A\vec{x} = \vec{b}$ can yield inconsistencies; cols of A don't span $\mathbb{R}^4$)

1g. Is T *one-to-one*? Explain in terms of the definition or any equivalent statement about A.

NO for example, $T\left(\begin{bmatrix} 3 \\ -3.5 \\ 0 \end{bmatrix}\right) = T\left(\begin{bmatrix} -2 \\ 1/2 \\ 1 \end{bmatrix}\right)$ (they both equal $\begin{bmatrix} 10 \\ -1 \\ 1 \\ 2 \end{bmatrix}$) yet $\begin{bmatrix} 3 \\ -3.5 \\ 0 \end{bmatrix} \neq \begin{bmatrix} -2 \\ 1/2 \\ 1 \end{bmatrix}$.

or, $T(\vec{x}) = \vec{0} \not\Rightarrow \vec{x} = \vec{0}$ since A has free variables (so $A\vec{x} = \vec{0}$ has ∞-many solutions)

2. Now let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + 3x_2 + 7 \\ x_1x_2 \end{bmatrix}$.

2A. Find the following:

$$T(\mathbf{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2+0+7 \\ 1 \cdot 0 \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \end{bmatrix} \quad T(\mathbf{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 10 \\ 0 \end{bmatrix} \quad T(\mathbf{e}_1 + \mathbf{e}_2) = T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2+3+7 \\ 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 12 \\ 1 \end{bmatrix}$$

2B. Use the results from (2A) to explain why T is *not* a linear transformation.

note that $T(\vec{e}_1) + T(\vec{e}_2) = \begin{bmatrix} 9 \\ 0 \end{bmatrix} + \begin{bmatrix} 10 \\ 0 \end{bmatrix} = \begin{bmatrix} 19 \\ 0 \end{bmatrix}$

while $T(\vec{e}_1 + \vec{e}_2) = \begin{bmatrix} 12 \\ 1 \end{bmatrix}$; since $\begin{bmatrix} 19 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 12 \\ 1 \end{bmatrix}$ we have $T(\vec{e}_1) + T(\vec{e}_2) \neq T(\vec{e}_1 + \vec{e}_2)$,

so that $T(\vec{u}) + T(\vec{v}) \neq T(\vec{u} + \vec{v})$ for all $\vec{u}, \vec{v} \in \mathbb{R}^2$, a violation of the 1st part of our defn of LINEAR transformation.