

NAME _____

I____ II____ III____ IV____ V____ VI____ VII____ VIII____ IX____ X____ TOTAL _____
(12) (15) (9) (8) (10) (9) (5) (12) (10) (10)

February 5
2009

Mathematics 309a
Abstract Algebra
Examination #1

Mr. Haines

(12) I. Give examples of:

A. a binary operation on the set $S = \{a, b\}$ that is not commutative. Explain why your operation is not commutative. Use this table to define the operation:

B. an Abelian group with 75 elements.

C. a finite subgroup of $\langle \mathbb{R}^*, \cdot \rangle$, (the non-zero real numbers under multiplication).

D. an infinite subgroup of $\langle \mathbb{R}^*, \cdot \rangle$ that is not a cyclic subgroup.

(15) II. Cyclic subgroups:

A. List the elements of the cyclic subgroup of $\langle \mathbb{Z}, + \rangle$ (the integers under addition) generated by 1.

B. List the elements of the cyclic subgroup of $\langle \mathbb{R}^*, \cdot \rangle$ (the non-zero real numbers under multiplication) generated by 2.

C. List the elements of the cyclic subgroup of $\langle \mathbb{C}^*, \cdot \rangle$ (the non-zero complex numbers under multiplication) generated by i .

D. List the elements of the cyclic subgroup of $\langle \mathbb{R}^*, \cdot \rangle$ generated by -2 .

E. List the elements of the cyclic subgroup of $\langle \mathbb{Z}_8, +_8 \rangle$ generated by 6.

(9) III. Draw the lattice diagram of subgroups of Z_{64} , the integers modulo 64, where the operation is $+_{64}$.

(8) IV. Find all solutions in \mathbb{C} (the complex numbers) to the equation $z^{16} - 1 = 0$.

(10) V. Let H be the subset of $M_2(\mathbb{R})$ consisting of all matrices of the form $\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$, where $b \in \mathbb{R}$. It is not hard to show that H is a group under matrix multiplication, denoted $*$, but you don't need to show it.

The cyclic subgroup K of H generated by $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is isomorphic to $\langle \mathbb{Z}, + \rangle$.

A. Compute the elements $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^3$ in K .

B. Give an isomorphism from $\langle \mathbb{Z}, + \rangle$ to $\langle K, * \rangle$. [You don't need to prove it is an isomorphism.]

(9) VI. If $\langle G, * \rangle$ is an abelian group with identity e , let $H = \{x \in G \mid x * x = e\}$. It can be shown that $\langle H, * \rangle$ is a subgroup of $\langle G, * \rangle$.

A. If $\langle G, * \rangle = \langle \mathbb{Z}_{32}, +_{32} \rangle$, list the elements of H .

B. If $\langle G, * \rangle = \langle \mathbb{Z}, + \rangle$, list the elements of H .

C. If $\langle G, * \rangle = \langle \mathbb{C}^*, \cdot \rangle$, list the elements of H .

(5) VII. Write $(-i)^{47}$ in the form $a + bi$

(12) VIII. Define a function $\phi: \mathbb{Z} \rightarrow \mathbb{Z}$ by the rule $\phi(n) = 5n$ for all integers n .

A. Prove that $\phi(m+n) = \phi(m) + \phi(n)$ for all integers m and n .

B. Prove that ϕ is one-to-one, i.e. if $\phi(m) = \phi(n)$ then $m = n$.

C. Prove that ϕ is not onto by giving an element of \mathbb{Z} that is not a value of ϕ .

D. Is ϕ an isomorphism? Explain why or why not.

(10) IX. Complete these two tables to give operations for two non-isomorphic groups $\langle \{e, a, b, c\}, * \rangle$ and $\langle \{e, a, b, c\}, *' \rangle$. Assume that e is the identity for each group.

A.

*	e	a	b	c
e				
a				
b				
c				

B.

*'	e	a	b	c
e				
a				
b				
c				

(10) X. TRUE OR FALSE? (Don't guess! The number of incorrect responses will be subtracted from the number of correct ones. Thus, random guessing earns you no points at all.)

_____ 1. The set of all real numbers is closed under multiplication.

_____ 2. Every group is Abelian.

_____ 3. Any two groups with 4 elements are isomorphic.

_____ 4. The empty set is an example of a group.

_____ 5. Every cyclic group has a generator.

_____ 6. All groups are cyclic.

_____ 7. Every cyclic group is Abelian.

_____ 8. Every infinite cyclic group is isomorphic to the integers under addition.

_____ 9. All groups are finite.

_____ 10. Every isomorphism of binary structures must be one-to-one and onto.