MATH 205 EXAM 1 FEBRUARY 5, 2008

NAME: KEY

YOUR GRADE IS BASED ON CORRECTNESS, COMPLETENESS, AND CLARITY ON EACH EXERCISE. YOU MAY USE A CALCULATOR, BUT NO NOTES, BOOKS, OR OTHER STUDENTS. GOOD LUCK!

1.) (15 pts.) Consider the system of equations

$$\begin{array}{rcrcrcrcl} 6x_1 & - & 2x_2 & & & + & 13x_4 & = & 52 \\ -5x_1 & + & x_2 & + & 5x_3 & & & = & 12 \\ & & 8x_2 & - & x_3 & - & 11x_4 & = & -7. \end{array}$$

a.) (5 pts.) Write the system as a vector equation.

$$x_1\begin{bmatrix} 6 \\ -5 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} -2 \\ 1 \\ 8 \end{bmatrix} + x_3\begin{bmatrix} 0 \\ 5 \\ -1 \end{bmatrix} + x_4\begin{bmatrix} 13 \\ 0 \\ -11 \end{bmatrix} = \begin{bmatrix} 52 \\ 12 \\ -7 \end{bmatrix}$$

b.) (5 pts.) Write the system as a matrix equation.

$$\begin{bmatrix} 6 & -2 & 0 & 13 \\ -5 & 1 & 5 & 0 \\ 0 & 8 & -1 & -11 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 52 \\ 12 \\ -7 \end{bmatrix}$$

c.) (5 pts.) Write the coefficient matrix of the system.

$$\begin{bmatrix} 6 & -2 & 0 & 13 \\ -5 & 1 & 5 & 0 \\ 0 & 8 & -1 & -11 \end{bmatrix}$$

2.) (15 pts.) Given matrices $A = \begin{bmatrix} 7 & 35 & -28 \\ 3 & 16 & -9 \\ -2 & -12 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 5 & -4 \\ 3 & 16 & -9 \\ -2 & -12 & 3 \end{bmatrix}$, complete the following.

a.) (3 pts.) State which elementary row operation transforms A into B.

Multiply Row 1 by $\frac{1}{7}$

b.) (7 pts.) *BY HAND*, perform the row operations that transform B into reduced echelon form. (Continue onto the back of this page if you need more space.)

$$\begin{bmatrix} 1 & 5 & -4 \\ 3 & 16 & -9 \\ -2 & -12 & 3 \end{bmatrix} \xrightarrow[\text{Mult R2 by 2, add to R3}]{\text{Mult. R1 by -3, add to R2}} \begin{bmatrix} 1 & 5 & -4 \\ 0 & 1 & 3 \\ 0 & -2 & -5 \end{bmatrix} \longrightarrow$$

$$\xrightarrow{\text{Mult. row 2 by 2, add to R3}} \begin{bmatrix} 1 & 5 & -4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[\text{Mult. row 3 by -3, add to R2}]{\text{Mult R3 by 4, add to R1}}$$

$$\begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{Mult } R_2 \text{ by -5, add to R1}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

c.) (5 pts.) Is the system $A\mathbf{x} = \mathbf{b}$ consistent for every vector $\mathbf{b}$ in $\mathbb{R}^3$? Explain.

Yes: there is a pivot in each of the 3 rows. (Th. 4)

3.) (15 pts.)

a.) (5 pts.) Construct a 4×4 matrix, not in echelon form, whose columns span $\mathbb{R}^4$. Explain how you know the columns span $\mathbb{R}^4$.

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Not in echelon form, but clearly can be rearranged to I_4, which has pivots in each of its 4 rows and therefore spans $\mathbb{R}^4$.

b.) (5 pts.) Suppose a system of linear equations has a 2×4 augmented matrix whose fourth column is a pivot column. Is the system consistent? Why or why not?

No: the last column represents the RHS of the system of equations. A pivot there means

$0x_1 + 0x_2 + 0x_3 = 1$, which is impossible.

c.) (5 pts.) Must a homogeneous system of equations always be consistent? Why or why not?

Yes: there is always the trivial solution,

$\vec{x} = \vec{0}$.

4.) (15 pts.)

a.) (5 pts.) What is a *quick* (meaning: no math needed) way to tell that the vectors $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 41 \\ 13 \end{bmatrix}$, and $\begin{bmatrix} 23 \\ 2 \end{bmatrix}$ are linearly dependent?

There are more than two vectors, each with only 2 entries.

In other words: these are 3 vectors in $\mathbb{R}^2$. There can be no more than 2 vectors from $\mathbb{R}^2$ in a linearly independent set.

b.) (5 pts.) Show that $\begin{bmatrix} 0 \\ 14 \\ 4 \end{bmatrix}$ is in Span$\left\{ \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} \right\}$.

$$\begin{bmatrix} 3 & 2 & 0 \\ 4 & -2 & 14 \\ 5 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{so} \quad 2\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} - 3\begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 14 \\ 4 \end{bmatrix},$$

that is, $\begin{bmatrix} 0 \\ 14 \\ 4 \end{bmatrix}$ is a linear combination of the other two vectors.

c.) (5 pts.) How many pivot columns must a 6×3 matrix have if its columns are linearly independent? Why?

3. There needs to be a pivot in <u>each</u> column, therefore 3 pivots.

5.) (15 pts.)

a.) (5 pts.) Determine whether the system below has a nontrivial solution.

$$\begin{array}{rcrcrcl} x_1 & + & 3x_2 & - & 5x_3 & = & 0 \\ x_1 & + & 4x_2 & - & 8x_3 & = & 0 \\ -3x_1 & - & 7x_2 & + & 9x_3 & = & 0 \end{array}$$

$$\begin{bmatrix} 1 & 3 & -5 \\ 1 & 4 & -8 \\ -3 & -7 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$ therefore any vector of the form

$\vec{x} = x_3 \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}$ solves the system. In other words: there is a nontrivial solution.

b.) (5 pts.) Suppose the solution set of a certain system of linear equations can be described as $x_1 = 3x_4$, $x_2 = 8 + x_4$, $x_3 = 2 - 5x_4$, and x_4 free. Use parametric vector form to describe this set as a "line" in $\mathbb{R}^4$.

$\vec{x} = x_4 \begin{bmatrix} 3 \\ 1 \\ -5 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 8 \\ 2 \\ 0 \end{bmatrix}$ This is a line in $\mathbb{R}^4$, parallel to the line of all multiples of $(3, 1, -5, 1)$, and passing through $(0, 8, 2, 0)$.

c.) (5 pts.) Let $T(\mathbf{e}_1) = \begin{bmatrix} 4 \\ -3 \\ -1 \end{bmatrix}$ and $T(\mathbf{e}_2) = \begin{bmatrix} -6 \\ -2 \\ 7 \end{bmatrix}$. Compute $T(2, -1)$.

$$T\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right) = T(2\vec{e}_1 + (-1)\vec{e}_2) = 2T(\vec{e}_1) - T(\vec{e}_2)$$

$$= 2\begin{bmatrix} 4 \\ -3 \\ -1 \end{bmatrix} - \begin{bmatrix} -6 \\ -2 \\ 7 \end{bmatrix} = \begin{bmatrix} 8 \\ -6 \\ -2 \end{bmatrix} + \begin{bmatrix} 6 \\ 2 \\ -7 \end{bmatrix} = \begin{bmatrix} 14 \\ -4 \\ -9 \end{bmatrix}$$

6.) (15 pts.) For the parts below, consider $\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -6x_1 + 4x_2 + x_3 \\ 5x_1 - 8x_2 + 2x_3 \end{bmatrix}$.

a.) (5 pts.) Fill in the missing entries of the matrix, assuming that the equation holds for all values of the variables. Explain how you know what each entry should be.

$\begin{bmatrix} -6 & 4 & 1 \\ 5 & -8 & 2 \end{bmatrix}$: This matrix, multiplied by any vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, results in the RHS shown.

b.) (5 pts.) Let A be the matrix you found in part (a). Is the transformation $T(\mathbf{x}) = A\mathbf{x}$ one-to-one? Explain your reasoning.

$A \sim \begin{bmatrix} 1 & 0 & -.57\ldots \\ 0 & 1 & -.61\ldots \end{bmatrix}$ A is a coefficient matrix having one column with no pivot, therefore one free variable. So $T(\vec{x})$ is not 1-1.

c.) (5 pts.) Let A again be the matrix you found in part (a). Is the transformation $T(\mathbf{x}) = A\mathbf{x}$ onto? Explain your reasoning.

Yes: there is one pivot per row, therefore $T(\vec{x})$ is onto $\mathbb{R}^2$.

7.) (10 pts.) A dietician is planning a meal that supplies certain quantities of vitamin C, calcium, and magnesium. Three foods will be used, their quantities measured in appropriate units. The nutrients supplied by these foods and the dietary requirements are given here.

	Milligrams (mg) of Nutrients per Unit of Food			
Nutrient	Food 1	Food 2	Food 3	Total Nutrients Required (mg)
Vitamin C	10	20	20	100
Calcium	50	40	10	300
Magnesium	30	10	40	200

Write a vector equation for this problem. State what the variables represent, and then solve the equations.

$$x_1 \begin{bmatrix} 10 \\ 50 \\ 30 \end{bmatrix} + x_2 \begin{bmatrix} 20 \\ 40 \\ 10 \end{bmatrix} + x_3 \begin{bmatrix} 20 \\ 10 \\ 40 \end{bmatrix} = \begin{bmatrix} 100 \\ 300 \\ 200 \end{bmatrix}$$

x_1, x_2, x_3 : mg of Foods 1, 2, 3, respectively in final mix.

$$\begin{bmatrix} 10 & 20 & 20 & 100 \\ 50 & 40 & 10 & 300 \\ 30 & 10 & 40 & 200 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 50/11 \\ 0 & 1 & 0 & 50/33 \\ 0 & 0 & 1 & 40/33 \end{bmatrix}$$

Let $x_1 = \frac{50}{11}$ mg of Food 1 $(4.\overline{54}$ mg$)$

$x_2 = \frac{50}{33}$ mg of Food 2 $(1.\overline{51}$ mg$)$

$x_3 = \frac{40}{33}$ mg of Food 3 $(1.\overline{21}$ mg$)$