NAME:

**Instruction:** Read each question carefully. Explain ALL your work and give reasons to support your answers.

*Advice:* DON’T spend too much time on a single problem.

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1.(16 pts.) (a) Consider the region bounded by the graph of \( y = 2x - x^2 \) and the graph of \( y = x^4 \). Find the exact area of the region. [Make a sketch of the region first.]

As shown in the figure above. The two curves intersect at two points, one at the origin and the other at \((1, 1)\). On a typical slice, the curve \( y = 2x - x^2 \) is above the curve \( y = x^4 \). Therefore, the area of the bounded region is given by

\[
A = \int_0^1 (2x - x^2) - x^4 \, dx
= x^2 - \frac{x^3}{3} - \frac{x^5}{5} \bigg|_0^1
= (1 - \frac{1}{3} - \frac{1}{5}) - 0 = \frac{7}{15}.
\]

(4 pts.) (b) Write (DO NOT evaluate) a definite integral representing the arc-length of the path given by \( y = 2x - x^2 \) from the origin \((0,0)\) to the point \((1,1)\).

Since \( y = 2x - x^2 \), \( y' = 2 - 2x \) and \( (y')^2 = 4 - 8x + 4x^2 \). The length of the path from \((0, 0)\) to \((1, 1)\) is given by

\[
L = \int_0^1 \sqrt{1 + (4 - 8x + 4x^2)} \, dx = \int_0^1 \sqrt{5 - 8x + 4x^2} \, dx.
\]
2. (12 pts.) (a) Consider a function \( f \) given by the following table.

\[
\begin{array}{c|ccccc}
  x & 2 & 2.5 & 3 & 3.5 & 4 \\
  f(x) & -1 & 5 & 2 & -2 & 3 \\
\end{array}
\]

Find \( T_4, M_2 \) using the trapezoidal and the mid-point rules respectively for the definite integral \( \int_2^4 f(x) \, dx \).

For \( M \) – 2, \( \Delta x = 1 \) whereas \( \Delta x = 0.5 \) when computing \( T_4 \). We have

\[
M_2 = 5 \cdot 1 + (-2) \cdot 1 = 3.
\]

The Left Hand Sum \( L_4 = [(-1) + 5 + 2 + (-2)] \cdot 0.5 = 2 \) and the Right Hand Sum \( R_4 = [5 + 2 + (-2) + 3] \cdot 0.5 = 4 \). It follows that

\[
T_4 = \frac{L_4 + R_4}{2} = 3.
\]

(8 pts.) (b) Evaluate the indefinite integral

\[
\int (e^x + \frac{1}{x}) \cdot \sin(e^x + \ln x) \, dx.
\]

Let \( w = e^x + \ln x \). Then \( dw = e^x + \frac{1}{x} \) so that

\[
\int (e^x + \frac{1}{x}) \cdot \sin(e^x + \ln x) \, dx = \int \sin w \, dw
\]

\[
= - \cos w + C
\]

\[
= - \cos(e^x + \ln x) + C.
\]
3. (20 pts.) The region bounded by the graph of \( y = \sqrt{x} \), the \( x \)-axis, and the line \( x = 2 \) is revolved about the \( y \)-axis. Find the exact volume of the resulting solid of revolution. [Sketch a picture of the region and the solid.]

As shown in the figure, a typical slice is a washer of thickness \( \Delta y \). When \( x = 2 \), the point on the curve of \( y = \sqrt{x} \) has \( y \)-coordinate equal to \( \sqrt{2} \). At any height \( y \), the surface area of the washer is \( \pi(2^2 - (y^2)^2) \) because the radius of the inner circle is given by the \( x \)-coordinate of the point on the curve of \( y = \sqrt{x} \).

Thus, the volume of the resulting solid is given by

\[
V = \int_0^{\sqrt{2}} \pi (2^2 - (y^2)^2) \, dy
= \pi \int_0^{\sqrt{2}} (4 - y^4) \, dy
= \pi \left[ 4y - \frac{y^5}{5} \right]_0^{\sqrt{2}}
= \sqrt{2} \pi \left[ 4 - \frac{4}{5} \right]
= \frac{16\sqrt{2}\pi}{5}
\]
4. (10 pts.) (a) Evaluate the indefinite integral
\[ \int \frac{2x^3 - x^2}{2x^2 - x - 3} \, dx. \]

First use long division to obtain that
\[ 2x^3 - x^2 = x(2x^2 - x - 3) + 3x. \]

Thus,
\[ \int \frac{2x^3 - x^2}{2x^2 - x - 3} \, dx = \int x + \frac{3x}{2x^2 - x - 3} \, dx \]
\[ = \frac{x^2}{2} + \int \frac{3x}{(x + 1)(2x - 3)} \, dx. \]

Using the method of partial fraction, we write
\[ \frac{3x}{(x + 1)(2x - 3)} = \frac{A}{x + 1} + \frac{B}{2x - 3} \]
and so
\[ 3x = A(2x - 3) + B(x + 1). \]

By evaluating the left hand side at \( x = -1 \) and at \( x = \frac{3}{2} \), we find that \( A = \frac{3}{5} \) and \( B = \frac{9}{5} \). Now,
\[ \int \frac{2x^3 - x^2}{2x^2 - x - 3} \, dx = \frac{x^2}{2} + \frac{3x}{5} \int \frac{dx}{x + 1} + \frac{9}{10} \int \frac{dx}{2x - 3} \]
\[ = \frac{x^2}{2} + \frac{3}{5} \ln |x + 1| + \frac{9}{10} \ln |2x - 3| + C. \]

(10 pts.) (b) Evaluate the definite integral
\[ \int_1^e \frac{\ln x}{x^2} \, dx. \]

Let \( u = \ln x \) and \( dv = x^{-2} \, dx \). Thus, \( du = \frac{dx}{x} \) and \( v = -\frac{1}{x} \). It follows from the technique of integration by parts that
\[ \int_1^e \frac{\ln x}{x^2} \, dx = -\frac{\ln x}{x} \bigg|_1^e - \int_1^e \left( -\frac{1}{x} \right) \cdot \frac{1}{x} \, dx \]
\[ = -\frac{\ln x}{x} \bigg|_1^e + \int_1^e x^{-2} \, dx \]
\[ = -\frac{\ln x}{x} \bigg|_1^e + \frac{x^{-1}}{-1} \bigg|_1^e \]
\[ = -\frac{1}{x} - \ln x \bigg|_1^e = 1 - \frac{2}{e}. \]
5. Consider the initial value problem
\[
\frac{dy}{dx} = xy^2
\]
with \(y(0) = 1\).

(10 pts.) (a) Solve the differential equation using the method of separation of variables.

Using the method of separating the variables, we have \( \frac{dy}{y^2} = x \, dx \).

Thus,
\[
\int \frac{dy}{y^2} = \int x \, dx
\]
which yields
\[
-\frac{1}{y} = \frac{x^2}{2} + C.
\]
Since \(y(0) = 1\), we have \(-1 = 0 + C\), or \(C = -1\). Solving for \(y\), we get
\[
y = \frac{2}{2 - x^2}.
\]

(10 pts.) (b) Estimate the value \(y(1)\) (when \(x = 1\)) of the solution using Euler’s method with two steps with initial point \((0, 1)\). DO THIS BY HAND and show all your steps.

First note that using two steps from \(x = 0\) to \(x = 1\) implies that the step size is \(\Delta x = 0.5\). With the initial point \((0, 1)\),
\[
y_{0.5} = y_0 + y'(x_0, y_0) \cdot \Delta x
\]
\[
= 1 + 0 \cdot 0.5 = 1.
\]

Next,
\[
y_1 = y_{0.5} + y'(x_{0.5}, y_{0.5}) \cdot \Delta x
\]
\[
= y_{0.5} + (x_{0.5}) (y_{0.5})^2 \cdot \Delta x
\]
\[
= 1 + (0.5)(1)^2 \cdot (0.5) = 1.25.
\]
Thus, the Euler’s method yields the approximation \(y(1) \approx 1.25\).