NAME: Solutions

Math 205 - Exam 1 - February 3, 2006

1. (8 pts.) Describe all solutions of $A\mathbf{x} = \mathbf{0}$ in parametric vector form, where the matrix A is

$$\begin{bmatrix} 1 & -4 & -2 & 0 \\ 2 & -8 & 1 & 3 \\ 0 & 1 & -1 & 3 \end{bmatrix}.$$

How can these solutions be described geometrically?

Look at the augmented matrix $[A \ \vec{0}]$ and row reduce to reduced echelon form.

$$\begin{bmatrix} 1 & -4 & -2 & 0 & 0 \\ 2 & -8 & 1 & 3 & 0 \\ 0 & 1 & -1 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & -2 & 0 & 0 \\ 0 & 0 & 3 & 3 & 0 \\ 0 & 1 & -1 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & -2 & 0 & 0 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 0 & 3 & 3 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -4 & -2 & 0 & 0 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 0 & 2 & 0 \\ 0 & 1 & 0 & 4 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 18 & 0 \\ 0 & 1 & 0 & 4 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

So
$$\left.\begin{aligned} x_1 + 18x_4 &= 0 \\ x_2 + 4x_4 &= 0 \\ x_3 + x_4 &= 0 \\ x_4 \text{ free.} & \end{aligned}\right\} \longrightarrow \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -18x_4 \\ -4x_4 \\ -x_4 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} -18 \\ -4 \\ -1 \\ 1 \end{bmatrix}.$$

The solutions are all pts on the line through the origin and $(-18, -4, -1, 1)$

2. (8 pts. each) Consider the transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$, given by $T(\mathbf{x}) = A\mathbf{x}$, where A is row equivalent to

$$\begin{bmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}.$$

(a) Does T map $\mathbb{R}^4$ onto $\mathbb{R}^3$? Explain.

(b) Is T one-to-one? Explain.

a) T is onto iff $A\vec{x} = \vec{b}$ has a solution for all $\vec{b}$ in $\mathbb{R}^3$.
Since A has a pivot in each row, by the "TFAE Thm" in section 1.4, $A\vec{x} = \vec{b}$ always has a solution, so T is onto.

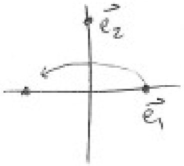
b) T is 1-1 iff $A\vec{x} = \vec{0}$ has a unique solution.
Since A has 4 columns and 3 rows, there are at most 3 pivots (there are exactly 3 pivots, actually), so in $A\vec{x} = \vec{0}$, we have a free variable, so T is not 1-1.

$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$

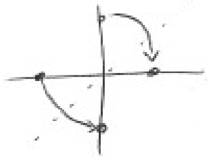
3. (8 pts.) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that first reflect points through the vertical x_2-axis and then reflects points through the line $x_2 = x_1$. Find the standard matrix of T.

first reflection

$\vec{e}_1 \xrightarrow{\text{reflect } 1} \begin{bmatrix} -1 \\ 0 \end{bmatrix} \xrightarrow{\text{reflect } 2} \begin{bmatrix} 0 \\ -1 \end{bmatrix}. \quad \vec{e}_2 \xrightarrow{\text{reflect } 1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \xrightarrow{\text{reflect } 2} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$



second reflection



So $T(\vec{e}_1) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, $T(\vec{e}_2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

So $T(\vec{x}) = [T(\vec{e}_1) \; T(\vec{e}_2)]\vec{x}$

$= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\vec{x}$, so the std matrix is $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

4. (4 pts. each) Consider the linear system

$$\begin{aligned} x_1 + hx_2 &= 2 \\ 3x_1 + 6x_2 &= k \end{aligned}$$

For what value(s) of h and k (if any) does this system have

(a) no solutions?

(b) a unique solution?

(c) infinitely many solutions?

$\begin{bmatrix} 1 & h & 2 \\ 3 & 6 & k \end{bmatrix} \sim \begin{bmatrix} 1 & h & 2 \\ 0 & 6-3h & k-6 \end{bmatrix}.$

a) No solutions means the last column has a pivot.

So $6-3h=0$ (so 2nd column does <u>not</u> have a pivot)

and $k-6 \neq 0$.

$\to h=2, \; k \neq 6.$

b) Unique solution means the system is consistent with no free vars.

ie, pivots in columns 1 and 2. $\to 6-3h \neq 0$, so $h \neq 2$.

(k is anything)

c) Infinitely many solutions means consistent with free vars.

So no pivot in columns 2 or 3.

$\to 6-3h = 0, \; k-6=0$, so $h=2, \; k=6$.

5. (6 pts. each) Let $A = \begin{bmatrix} 3 & 1 \\ -4 & -2 \end{bmatrix}$.

(a) Calculate A^{-1}.

$= \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, provided $ad-bc \neq 0$.

$ad-bc = 3(-2)-(-4)1 = -2$.

So $A^{-1} = \frac{1}{-2}\begin{bmatrix} -2 & -1 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1/2 \\ -2 & -3/2 \end{bmatrix}$.

(b) Using A^{-1}, find all solutions to the matrix equation $A\mathbf{x} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$, if any exist.

$A\vec{x} = \vec{b} \rightarrow \vec{x} = A^{-1}\vec{b}$, provided A is invertible (which it is here).

So $\vec{x} = \begin{bmatrix} 1 & 1/2 \\ -2 & -3/2 \end{bmatrix}\begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$. Since A is invertible, this is the unique solution.

6. (3 pts. each) Answer the following as being true or false. You do **not** need to justify your answers.

(a) If the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent, then $\mathbf{b}$ is not in the set spanned by the columns of A.

True. (Because $A\vec{x} = x_1\vec{a}_1 + \cdots + x_n\vec{a}_n$.)

(b) The columns of any 5×4 matrix are dependent.

False. (Look at $\begin{bmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \\ 0&0&0&0 \end{bmatrix}$ for instance.)

(c) Not every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a matrix transformation.

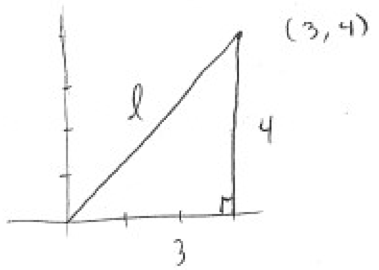
False. (The matrix A is given by $[T(\vec{e}_1) \cdots T(\vec{e}_n)]$.

(d) For any $n \times n$ matrices A, B, and C, if $AB = AC$, then $B = C$.

False. (~~Only~~ If A is invertible, this is true.
But in general, this is not necessarily true.)

7. (8 pts.) The *dot product* of two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, denoted by $\mathbf{u} \cdot \mathbf{v}$, is defined by $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T\mathbf{v}$. For $\mathbf{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, find $\mathbf{v} \cdot \mathbf{v}$. How does this compare to the distance from the point $(0,0)$ to the point $(3,4)$? (Hint: Draw a right triangle and use the Pythagorean theorem..)

$$\begin{bmatrix} 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 9+16 \end{bmatrix} = 25.$$



$$3^2 + 4^2 = l^2$$
$$25 = l^2$$
$$5 = l.$$

The distance is the square root of $\vec{v} \cdot \vec{v}$.

8. (8 pts.) Suppose A is a 3x4 matrix and $\mathbf{y}$ is a vector in $\mathbb{R}^3$ such that the equation $A\mathbf{x} = \mathbf{y}$ does *not* have a solution. Does there exist a vector $\mathbf{z}$ in R^3 such that the equation $A\mathbf{x} = \mathbf{z}$ has a unique solution? Explain your answer.

Since A is 3×4, there will have to be at least one free variable when solving $A\vec{x} = \vec{z}$. Therefore, if there are any solutions, there are infinitely many, and if there are none, then there are none. (In either case, we won't have a unique solution.)

9. (8 pts.) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a linearly dependent set of vectors in $\mathbb{R}^n$. Explain why the set $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ is linearly dependent.

If $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a linearly dependent set, then there are scalars c_1, c_2, c_3, not all 0, where $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$.

Now, $c_1 T(\vec{v}_1) + c_2 T(\vec{v}_2) + c_3 T(\vec{v}_3) = T(c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3)$ because T is a linear transform, and this is $T(\vec{0})$, which is $\vec{0}$. So we have a non-trivial linear combination of $T(\vec{v}_1), T(\vec{v}_2), T(\vec{v}_3)$ that equals $\vec{0}$, so these vectors are linearly dependent.

10. (8 pts.) Suppose A is an invertible $n \times n$ matrix. Explain, without using the invertible matrix theorem, why the columns of A are linearly independent.

If A is invertible, then by Thm 5 in section 2.2, $A\vec{x} = \vec{b}$ has a unique solution for any $\vec{b}$, so $A\vec{x} = \vec{0}$ has a unique solution, which means the columns of A are linearly independent.

OR:

If A invertible, then $A \sim I_n$, so A has n pivots, so in solving $A\vec{x} = \vec{0}$, there are no free variables, so $A\vec{x} = \vec{0}$ has a unique solution $(\vec{x} = \vec{0})$, so the columns are linearly independent.

1	2	3	4	5	6	7	8	9	10	TOTAL