

NAME: SOLUTIONS

Math 205 - Exam 1 - February 3, 2006

1. (8 pts.) Describe all solutions of $Ax = \vec{0}$ in parametric vector form, where the matrix A is

$$\begin{bmatrix} 1 & -4 & -2 & 0 \\ 2 & -8 & 1 & 3 \\ 0 & 1 & -1 & 3 \end{bmatrix}$$

How can these solutions be described geometrically?

Look at the augmented matrix $[A \ \vec{0}]$ and row reduce to reduced echelon form.

$$\begin{bmatrix} 1 & -4 & -2 & 0 & 0 \\ 2 & -8 & 1 & 3 & 0 \\ 0 & 1 & -1 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & -2 & 0 & 0 \\ 0 & 0 & 3 & 3 & 0 \\ 0 & 1 & -1 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & -2 & 0 & 0 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -4 & -2 & 0 & 0 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 0 & 2 & 0 \\ 0 & 1 & 0 & 4 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 18 & 0 \\ 0 & 1 & 0 & 4 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

So $\begin{cases} x_1 + 18x_4 = 0 \\ x_2 + 4x_4 = 0 \\ x_3 + x_4 = 0 \\ x_4 \text{ free.} \end{cases} \rightarrow \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -18x_4 \\ -4x_4 \\ -x_4 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} -18 \\ -4 \\ -1 \\ 1 \end{bmatrix}$

The solutions are all pts on the line through the origin and $(-18, -4, -1, 1)$.

2. (8 pts. each) Consider the transformation $T: \mathbb{R}^4 \rightarrow \mathbb{R}^3$, given by $T(\vec{x}) = A\vec{x}$, where A is row equivalent to

$$\begin{bmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

- (a) Does T map \mathbb{R}^4 onto \mathbb{R}^3 ? Explain.
 (b) Is T one-to-one? Explain.

a) T is onto iff $A\vec{x} = \vec{b}$ has a solution for all \vec{b} in \mathbb{R}^3 .

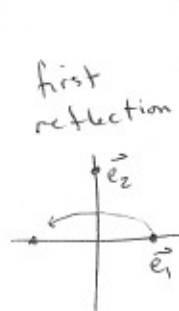
Since A has a pivot in each row, by the "TFAE Thm" in section 1.4, $A\vec{x} = \vec{b}$ always has a solution, so T is onto.

b) T is 1-1 iff $A\vec{x} = \vec{0}$ has a unique solution.

Since A has 4 columns and 3 rows, there are at most 3 pivots (there are exactly 3 pivots, actually), so in $A\vec{x} = \vec{0}$, we have a free variable, so T is not 1-1.

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

3. (8 pts.) Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear transformation that first reflect points through the vertical x_2 -axis and then reflects points through the line $x_2 = x_1$. Find the standard matrix of T .



$$\vec{e}_1 \xrightarrow{\text{reflect } 1} \begin{bmatrix} -1 \\ 0 \end{bmatrix} \xrightarrow{\text{reflect } 2} \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad \vec{e}_2 \xrightarrow{\text{reflect } 1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \xrightarrow{\text{reflect } 2} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

second reflection



$$\text{So } T(\vec{e}_1) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad T(\vec{e}_2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$$\text{So } T(\vec{x}) = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{bmatrix} \vec{x}$$

$$= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \vec{x}, \quad \text{so the std matrix is } \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

4. (4 pts. each) Consider the linear system

$$\begin{aligned} x_1 + hx_2 &= 2 \\ 3x_1 + 6x_2 &= k \end{aligned}$$

For what value(s) of h and k (if any) does this system have

- no solutions?
- a unique solution?
- infinitely many solutions?

$$\begin{bmatrix} 1 & h & 2 \\ 3 & 6 & k \end{bmatrix} \sim \begin{bmatrix} 1 & h & 2 \\ 0 & 6-3h & k-6 \end{bmatrix}.$$

a) No solutions means the last column has a pivot.

So $6-3h=0$ (so 2nd column does not have a pivot) and $k-6 \neq 0$.

$$\rightarrow h=2, k \neq 6.$$

b) Unique solution means the system is consistent with no free vars. ie, pivots in columns 1 and 2. $\rightarrow 6-3h \neq 0$, so $h \neq 2$. (k is anything)

c) Infinitely many solutions means consistent with free vars.

So no pivot in columns 2 or 3.

$$\rightarrow 6-3h \stackrel{!}{=} 0, \quad k-6=0, \quad \text{so } h=2, k=6.$$

5. (6 pts. each) Let $A = \begin{bmatrix} 3 & 1 \\ -4 & -2 \end{bmatrix}$.

(a) Calculate A^{-1} .

$$= \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, \text{ provided } ad-bc \neq 0.$$

$$ad-bc = 3(-2) - (-4)(1) = -2.$$

$$\text{So } A^{-1} = \frac{1}{-2} \begin{bmatrix} -2 & -1 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1/2 \\ -2 & -3/2 \end{bmatrix}.$$

(b) Using A^{-1} , find all solutions to the matrix equation $Ax = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$, if any exist.

$$A\vec{x} = \vec{b} \rightarrow \vec{x} = A^{-1}\vec{b}, \text{ provided } A \text{ is invertible (which it is here).}$$

$$\text{So } \vec{x} = \begin{bmatrix} 1 & 1/2 \\ -2 & -3/2 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}. \text{ Since } A \text{ is invertible, this is the unique solution.}$$

6. (3 pts. each) Answer the following as being true or false. You do **not** need to justify your answers.

(a) If the equation $Ax = b$ is inconsistent, then b is not in the set spanned by the columns of A .

True. (Because $A\vec{x} = x_1\vec{a}_1 + \dots + x_n\vec{a}_n$.)

(b) The columns of any 5×4 matrix are dependent.

False. (Look at $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ for instance.)

(c) Not every linear transformation from \mathbb{R}^n to \mathbb{R}^m is a matrix transformation.

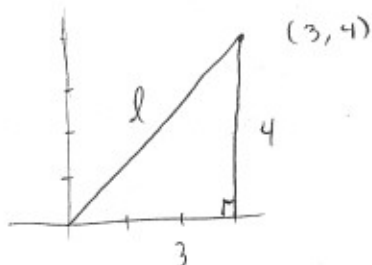
False. (The matrix A is given by $[T(\vec{e}_1) \dots T(\vec{e}_n)]$.)

(d) For any $n \times n$ matrices A , B , and C , if $AB = AC$, then $B = C$.

False. (Only if A is invertible, this is true. But in general, this is not necessarily true.)

7. (8 pts.) The dot product of two vectors \mathbf{u} and \mathbf{v} in \mathbb{R}^n , denoted by $\mathbf{u} \cdot \mathbf{v}$, is defined by $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v}$. For $\mathbf{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, find $\mathbf{v} \cdot \mathbf{v}$. How does this compare to the distance from the point $(0,0)$ to the point $(3,4)$? (Hint: Draw a right triangle and use the Pythagorean theorem.)

$$\begin{bmatrix} 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 9+16 \end{bmatrix} = 25.$$



$$3^2 + 4^2 = l^2$$

$$25 = l^2$$

$$5 = l.$$

The distance is the square root of $\vec{v} \cdot \vec{v}$.

8. (8 pts.) Suppose A is a 3×4 matrix and \mathbf{y} is a vector in \mathbb{R}^3 such that the equation $A\mathbf{x} = \mathbf{y}$ does not have a solution. Does there exist a vector \mathbf{z} in \mathbb{R}^3 such that the equation $A\mathbf{x} = \mathbf{z}$ has a unique solution? Explain your answer.

Since A is 3×4 , there will have to be at least one free variable when solving $A\vec{x} = \vec{z}$. Therefore, if there are any solutions, there are infinitely many, and if there are none, then there are none. (In either case, we won't have a unique solution.)

