1. Integrate the following. If you use a formula from the table of integrals, indicate which formula you’re using. (5 pts. each)

(a) \( \int x^2 \cos(x^3) \, dx \)

\[ u = x^3, \quad du = 3x^2 \, dx \]

\[ = \frac{1}{3} \int \cos u \, du = \frac{1}{3} \sin u + C = \frac{1}{3} \sin(x^3) + C. \]

(b) \( \int_0^1 \arctan(x) \, dx \)

Parts. \( u = \arctan x \quad du = \frac{dx}{1+x^2} \quad v = x \)

\[ = x \arctan x \bigg|_0^1 - \int_0^1 \frac{x}{1+x^2} \, dx \]

\[ = \left( \frac{\pi}{4} \right) - \frac{1}{2} \left[ \ln(1) - \ln(1) \right] = \frac{\pi}{4} - \frac{1}{2} \ln(2) \]

(c) \( \int \frac{3x^2 + x + 3}{x^3 + 1} \, dx \)

Partial fractions.

\[ \frac{3x^2 + x + 3}{x(x^2 + 1)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1} \]

\[ 3x^2 + x + 3 = A(x^2 + 1) + (Bx + C)x \]

\[ 3x^2 + x + 3 = A(x^2 + 1) + Bx + Cx \]

\[ = 3 = A + B, \quad 1 = C, \quad 3 = A. \]

\[ \int \frac{3x^2 + x + 3}{x^3 + 1} \, dx = \int \left( \frac{3}{x} + \frac{1}{x^2 + 1} \right) \, dx = \frac{3}{x} \ln(x^2 + 1) + \arctan(x) + C. \]
(d) \[ \int_{1}^{e} \frac{2x+3}{(x+2)^2} \, dx \]

Let \( u = x + 2 \)

\[ du = dx \quad u(1) = 3, \quad u(e) = 5 \]

\[ \frac{u^3}{2} = \int \frac{2(u-3)+3}{u^2} \, du = \int \frac{2u-3}{u^2} \, du = \int \frac{2}{u} \, du - \frac{3}{u} \, du = 2\ln u - 3\ln u \]

\[ = (2\ln 5 - \frac{3}{5}) - (2\ln 3 + \frac{1}{3}) \]

(e) \[ \int \frac{e^{\sqrt{x}} \cos(\sqrt{x})}{\sqrt{x}} \, dx \]

Let \( u = \sqrt{x} \)

\[ du = \frac{1}{2\sqrt{x}} \, dx \]

\[ = 2 \int e^u \cos u \, du = 2 \left( \frac{1}{2^2} \cos u + \frac{1}{2} \sin u \right) + C \]

from table, formula 55

\[ = e^\sqrt{x} (\cos \sqrt{x} + \sin \sqrt{x}) + C \]

2. (8 pts.) Calculate the area of the region bounded by the function \( y = \sqrt{x} \), the \( x \)-axis, and the line \( y = x - 2 \). (Hint: Integrate in terms of \( y \))

Intersection when \( \sqrt{x} = x - 2 \), so \( x = (x-2)^2 \)

\[ x = x^2 - 4x + 4, \quad x^2 - 5x + 4 = 0 \]

\( (x-4)(x-1) = 0 \)

We need \( x \geq 2 \), so intersection occurs at \( x = 4 \). Point is \((4, 2)\).

\[ A = \int_{0}^{y} \text{(right fn)} - (\text{left fn}) \, dy \]

\[ = \int_{0}^{2} y^2 - y^2 \, dy = \int_{0}^{2} y^2 \, dy = \left[ \frac{y^3}{3} \right]_{0}^{2} = \left( \frac{8}{3} - 0 \right) - (0 + 0 - 0) \]

\[ = \frac{10}{3} \]
3. (5 pts.) Suppose \( f(x) \) is decreasing and concave up on the interval \([a, b]\). Put the following in order from smallest to largest: \( L_{20}, R_{20}, M_{20}, T_{20}, \int_{a}^{b} f(x) \, dx \).

\[
L_{20} \leq M_{20} \leq \int_{a}^{b} f(x) \, dx \leq T_{20} \leq R_{20}.
\]

4. Consider the IVP \( y' = t + y, \ y(0) = 1 \).

   (a) (5 pts.) By hand, using a time-step of 1/2, give an estimate for \( y(1) \).

   \[
y(0) = 1, \quad \Delta t = \frac{1}{2}.
   \]

   1st time step. \( \rightarrow \)

   \[
y(\frac{1}{2}) = y(0) + m(0) \cdot \Delta t = 1 + (0 + y(0)) \cdot \frac{1}{2} = 1 + (0 + 1) \cdot \frac{1}{2} = \frac{3}{2}.
   \]

   \[
y(1) = y\left(\frac{1}{2}\right) + m\left(\frac{1}{2}\right) \cdot \frac{1}{2} = \frac{3}{2} + m\left(\frac{1}{2} + y\left(\frac{1}{2}\right)\right) \cdot \frac{1}{2} = \frac{3}{2} + \left(\frac{1}{2} + \frac{3}{2}\right) \cdot \frac{1}{2} = \frac{5}{2}.
   \]

   (b) (3 pts.) Using a time step of .1, use your calculator to estimate \( y(1) \). Do you expect this to be an overestimate or underestimate? Explain.

   \[
y(1) \approx 3.18748 \text{ from calculator. The slope field shows}
   \]

   \[
   \text{the solution will be concave up, so this is an underestimate.}
   \]

   \[
   (A \text{ so, } y'' = (t + y)' = (t + y)' = (1 + y') = (1 + 1 + 1) \text{ since } y > 0, \ y'' > 0.}
   \]

5. (5 pts. each)

   (a) Solve the IVP \( y' = x^{2}e^{-y}, \ y(0) = 1 \).

   \[
   \frac{dy}{dx} = \frac{x^{2}}{e^{y}} \rightarrow \int e^{y} \, dy = \int x^{2} \, dx \rightarrow e^{y} = \frac{x^{3}}{3} + C. \rightarrow y = \ln \left| \frac{x^{3}}{3} + C \right|.
   \]

   \[
y(0) = 1 \rightarrow 1 = \ln \left| \frac{0}{3} + C \right| = \ln |C|, \text{ so } C = e \text{ works. (or } C = -e).\]

   \[
   \boxed{y(x) = \ln \left| \frac{x^{3}}{3} + e \right|}.
   \]

   (b) Find all solutions of the DE \( \frac{dy}{dt} = t/y^{2} \).

   \[
   \frac{dy}{dt} = \frac{t}{y^{2}} \rightarrow \int y^{2} \, dy = \int t \, dt \rightarrow \frac{y^{3}}{3} = \frac{t^{2}}{2} + C.
   \]

   \[
   \rightarrow y^{3} = \frac{3}{2} + C.
   \]

   \[
   \boxed{y = 2 \sqrt[3]{C} + C.}
   \]
6. (5 pts. each) Let $I = \int_1^4 \ln x \, dx$.

(a) Calculate $M_{10}$. (Use your calculator.) Explain why this will be an overestimate.

The graph of $y = \ln x$ is concave down, so $M_{10}$ overestimates.

$M_{10} = 2.544971$.

(b) Using the error bound theorem, how accurate (to six decimals) is $M_{10}$ as an estimate of $I$?

$|I - M_{10}| \leq \frac{k_2 (4-1)^5}{24 n^2} = \frac{1.27}{24 \cdot 100} = 0.011250$.

On $[1, 4]$, $|f''(x)| \leq 1$.

So $k_2 = 1$.

(c) Solve the integral $I$ and give the actual error (to six decimals) of $M_{10}$.

$I = \int_1^4 \ln x \, dx$

Let $u = \ln x$, $du = \frac{1}{x} \, dx$

$= \left[ \ln x \right]_1^4 - \int_1^4 \frac{1}{x} \, dx$

$= (4 \ln 4 - 1) - \int_1^4 \, dx$

$= (4 \ln 4) - 3 \approx 2.54517444$.

(d) Using the error bound theorem, how many subintervals would we need to use with Simpson's rule to guarantee accuracy to within 0.00001?

$f'''(x) = \frac{2}{x^3}$, $|f'''(x)| \leq 6$. On $[1, 4]$, $|f'''(x)| \leq 6$.

$|I - S_n| \leq \frac{(6)(4-1)^5}{180 n^4} \leq 0.00001$.

$n^4 \geq \frac{(6)(4-1)^5}{180 \cdot 0.00001} = 810000$.

So $n \geq 90$. 

$\boxed{n \geq 30}$. 

7. (8 pts.) Let \( f(x) = \frac{x^3}{2} + 5 \). Find the length of the curve \( y = f(x) \) from \( x = 0 \) to \( x = 3 \).

\[
\text{Arc length} = \int_0^3 \sqrt{\left(f'(x)\right)^2 + 1} \, dx
\]

\[
f(x) = \frac{x^3}{2} + 5,
\]

\[
f'(x) = \frac{3x^2}{2}
\]

\[
f'(x) = \frac{1}{3}x^2 + a = \sqrt{x}.
\]

\[
\begin{align*}
\int_0^3 \sqrt{\left(f'(x)\right)^2 + 1} \, dx &= \int_0^3 \sqrt{\frac{1}{3}x^2 + 1} \, dx \\
&= \int_0^3 \sqrt{x+1} \, dx \\
&= \int_0^3 \sqrt{u} \, du \\
&= \frac{2}{3} \int_0^3 u^{3/2} \, du \\
&= \frac{2}{3} \left[ \frac{2}{3} u^{3/2} \right]_1^4 \\
&= \frac{16}{3} - \frac{2}{3}
\end{align*}
\]

8. (8 pts.) Find the volume of the solid of revolution formed when the region bounded by the curve \( y = x^2 \) and the x-axis between \( x = 0 \) and \( x = 2 \) is rotated around the y-axis.

\[
\text{Volume} = \pi \int_0^2 \left(2^2 - x^2\right) \, dx
\]

\[
= \pi \int_0^2 \left(4 - x^2\right) \, dx
\]

\[
= \pi \left[ 4x - \frac{x^3}{3} \right]_0^2
\]

\[
= \pi (8) - \frac{8}{3}
\]

\[
= \frac{24\pi}{3}
\]
9. (8 pts.) Suppose that a tank in the shape of a circular cone, standing on its point, with height 10 feet and radius 5 feet (at the top) is partially filled with water (which weighs 62.4 pounds per cubic foot). If the water is 4 feet deep at the deepest point, how much work must be done to pump all of the water over the top rim of the tank? (NOTE: Just set up the integral - you do not need to solve it.)

Weight of slice = 62.4 \cdot \pi \frac{y^2}{4} \ dy.

Need to move slice from height y to height 10. \text{ Dist} = 10 - y.

So total work for a slice is 62.4 \pi \frac{y^2}{4} y (10 - y) \text{ ft-lbs}.

Total work from y = 0 to y = 4 is

\[ \int_{0}^{4} 62.4 \pi \frac{y^2}{4} (10 - y) \ dy \]