MATH106A CALCULUS II - PROF. P. WONG

EXAM I - FEBRUARY 2, 2007

NAME:

Instruction: Read each question carefully. Explain **ALL** your work and give reasons to support your answers.

Advice: DON'T spend too much time on a single problem.

Problems	**Maximum Score**	**Your Score**
1.	20	
2.	20	
3.	20	
4.	20	
5.	20	
Total	100	

1.(10 pts.)(a) Evaluate the definite integral

$$\int_0^{\pi/2} e^{\sin x} \cos x \ dx.$$

Let $u = \sin x$**. Then,** $du = \cos x \ dx$**. When** $x = 0, u = 0$ **and when** $x = \pi/2, u = 1$**. Thus, we have**

$$\int_0^{\pi/2} e^{\sin x} \cos x \ dx = \int_0^1 e^u \ du$$
$$= e^u \Big|_0^1 = e - 1.$$

(10 pts.)(b) Evaluate the indefinite integral

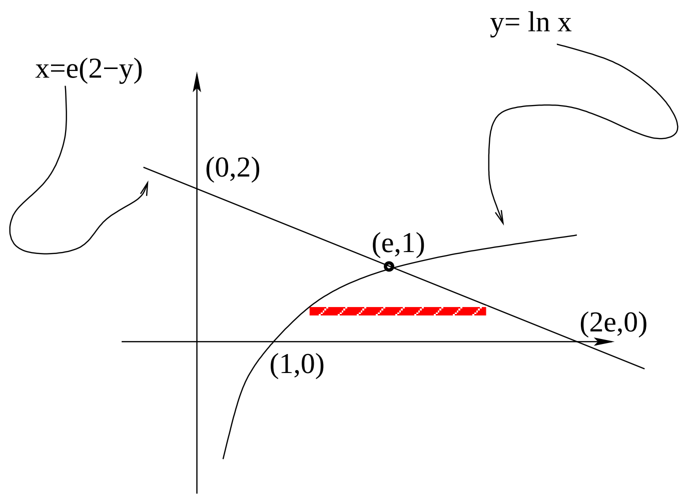
$$\int \frac{2x + 2x^3}{\sqrt{1+x^2}} \ dx.$$

Let $u = 1 + x^2$ **so that** $du = 2x \ dx$**. Now, the numerator** $2x + 2x^3 \ dx$ **becomes** $2x(1 + x^2)dx = u \ du$**. Thus,**

$$\int \frac{2x + 2x^3}{\sqrt{1+x^2}} \ dx = \int \frac{u}{\sqrt{u}} \ du$$
$$= \int u^{1/2} \ du$$
$$= \frac{u^{3/2}}{3/2} + C = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(1 + x^2)^{3/2} + C.$$

2.(20 pts.) Find the area of the region bounded by the curve $y = \ln x$, the x-axis, and the line $x = e(2-y)$. [Hint: sketch a picture of the region by determining the points of intersections among these curves and lines]

First we sketch the graphs fo these curves. They intersect at the point $(e, 1)$**.**



By choosing a horizontal slice, the x**-coordinate of the left endpoint of the slice is** e^y **(derived from the curve** $y = \ln x$**) while the** x**-coordinate of the right endpoint is** $e(2-y)$**. Thus, the area of this bounded region is given by**

$$\int_0^1 e(2-y) - e^y \; dy = e(2y - \frac{y^2}{2}) - e^y \Big|_0^1$$
$$= [e(2-\frac{1}{2}) - e] - [0 - e^0] = \frac{e}{2} + 1.$$

If you want to use vertical slices, you need to write the area as the sum of two definite integrals as follows.

$$\int_1^e \ln x \; dx + \int_e^{2e} 2 - \frac{x}{e} \; dx.$$

3. (10 pts.)(a) Consider a function f given by the following table.

x	2	2.5	3	3.5	4
$f(x)$	2	0	4	-2	1

Find T_4, M_2 using the trapezoidal and the mid-point rules respectively for the definite integral $\int_2^4 f(x)\,dx$.

First, we compute the Left-Hand Sum and the Right-Hand Sum. Note that $\Delta x = 0.5$**.**

$L_4 = (2 + 0 + 4 + (-2))(0.5) = 2 \qquad R_4 = (0 + 4 + (-2) + 1)(0.5) = 1.5$

so

$$T_4 = \frac{L_4 + R_4}{2} = 1.75.$$

Similarly, for the Mid-point Rule, we have only two subintervals each of which has length 1**, i.e.,** $\Delta x = 1$**. Now,**

$$M_2 = (f(2.5) + f(3.5))\Delta x = (0 + (-2))(1) = -2.$$

(10 pts.)(b) Recall that the error committed by using the trapezoid approximation T_n is less than or equal to $\frac{K_2 \cdot (b-a)^3}{12n^2}$ where $|f''(x)| \leq K_2$ for some constant K_2 over the interval $[a, b]$. Use this result to give an upper bound for the error committed by T_8 for

$$I = \int_0^1 e^{x^2}\,dx.$$

Here $f(x) = e^{x^2}$**. It follows that** $f'(x) = e^{x^2}2x$ **and** $f''(x) = (e^{x^2}2x)(2x) + (e^{x^2})(2)$**. Over the interval** $[0, 1]$**, we have**

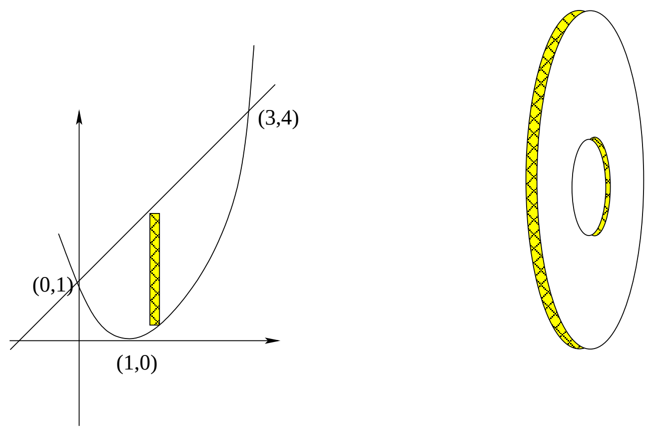
$$|f''(x)| \leq (e^1 2)(2) + e^1 2 = 6e$$

so we let $K_2 = 6e$**. It follows that**

$$|I - T_8| \leq \frac{6e(1-0)^3}{12 \cdot 8^2} = \frac{e}{128}.$$

4. Let R be the region bounded by the graph of $y = (x-1)^2$ and the line $y = x + 1$.

(15 pts.) Find the volume of the solid obtained from rotating the region around the x-axis.



The points of intersection A **and** B **are** $(0,1)$ **and** $(3,4)$ **respectively. A typical slice of this solid is a "washer" whose thickness is** Δx**. The radii are measured from the straight line** $y = x + 1$ **and from the curve** $y = (x-1)^2$ **to the** x**-axis respectively. Thus, the volume of the solid is given by**

$$V = \int_0^3 \pi(x+1)^2 - \pi[(x-1)^2]^2 \; dx$$
$$= \pi \int_0^3 (x^2 + 2x + 1) - (x^4 - 4x^3 + 6x^2 - 4x + 1) \; dx$$
$$= \pi \int_0^3 -x^4 + 4x^3 - 5x^2 + 6x \; dx = \pi - \frac{x^5}{5} + x^4 - 5\frac{x^3}{3} + 3x^2 \Big|_0^3 = \frac{72\pi}{5}.$$

(5 pts.) SET UP (**do not evaluate**) a definite integral representing the arc length of the portion of the curve $y = (x-1)^2$ from A to B where A and B are the points of intersection.

Here $f(x) = (x-1)^2$ **so that** $f'(x) = 2(x-1)$ **and** $[f'(x)]^2 = 4(x^2 - 2x + 1)$**. The length of the curve from** $A = (0,1)$ **to** $B = (3,4)$ **is given by**

$$L = \int_0^3 \sqrt{1 + [f'(x)]^2} \; dx = \int_0^3 \sqrt{1 + 4(x^2 - 2x + 1)} \; dx = \int_0^3 \sqrt{4x^2 - 8x + 5)} \; dx.$$

5. (10 pts.)(a) Consider the initial value problem

$$\frac{dy}{dx} = xy$$

with $y(0) = -2$.

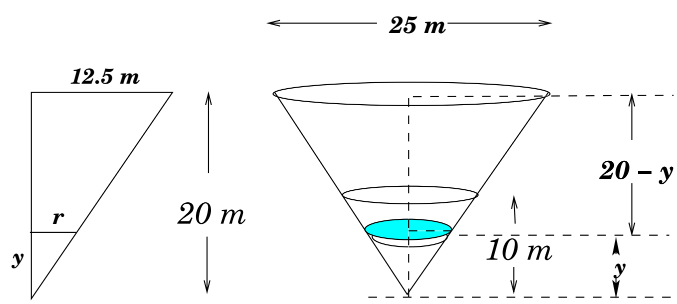
Estimate the value $y(1)$ (when $x = 1$) of the solution using Euler's method with two steps with initial point $(0, -2)$. DO THIS BY HAND and show all your steps.

There are two subintervals with $x_0 = 0, x_1 = 0.5$ **and** $x_2 = 1$**. By the Euler's method, we have** $y_1 = y_0 + f(x_0, y_0)\Delta x = (-2) + (x_0 y_0)(0.5) = -2$**. It follows that**

$$y_2 = y_1 + (x_1 y_1)\Delta x = (-2) + (0.5)(-2)(0.5) = -2.5.$$

Thus, $y(1) \approx -2.5$**.**

(10 pts.)(b) SET UP (**do not evaluate**) a definite integral representing the work done in pumping fluid from the cone-shaped tank in the figure to the rim. The fluid has density ρ (constant) and the depth of the fluid is 10m. [gravitational constant is g]



A typical layer of fluid has weight $\rho g \pi r^2 \Delta y$**. Using similar triangles,** $\frac{r}{y} = \frac{12.5}{20}$ **or** $r = \frac{5y}{8}$**. The distance to be lifted is** $20 - y$ **and the layer varies from height** $y = 0$ **to** $y = 10$**. Thus, the total work done is given by**

$$W = \int_0^{10} \rho g \pi \left(\frac{5y}{8}\right)^2 (20 - y)\, dy.$$