NAME: KEY

Math 106B - Exam 1 - February 2, 2007

INSTRUCTIONS: Show all of your work and circle your solutions. Cross out any unnecessary work.

1. (10 points each)

(a) $\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}}$

$u = \ln x$, $du = \frac{1}{x}\,dx$.

$x = e \to u = \ln e = 1$

$x = e^4 \to u = \ln e^4 = 4$.

$$= \int_{u=1}^{u=4} \frac{du}{\sqrt{u}} = \int_1^4 u^{-1/2}\,du = 2u^{1/2}\Big]_1^4 = 2\sqrt{4} - 2\sqrt{1} = 2\cdot 2 - 2\cdot 1 = \boxed{2.}$$

(b) $\int x(2x+1)^9\,dx$

$u = 2x+1$

$du = 2dx$, $dx = \frac{1}{2}du$ $\quad x = \frac{u-1}{2}$

$$= \frac{1}{2}\int\left(\frac{u-1}{2}\right)u^9\,du = \frac{1}{4}\int u^{10} - u^9\,du = \frac{1}{4}\left(\frac{u^{11}}{11} - \frac{u^{10}}{10}\right) + C$$

$$\boxed{= \frac{(2x+1)^{11}}{44} - \frac{(2x+1)^{10}}{40} + C.}$$

2. Let $I = \int_1^7 v(t)\, dt$, where particular values of the function $v(t)$ are given in the table below. (Note the endpoints of the integral!)

| $t$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $v(t)$ | 6.3 | 4.4 | 2.9 | 1.6 | 0.6 | -0.3 | -0.9 | -1.4 | -1.7 |

(a) (4 points) Suppose $v(t)$ gives the eastward velocity at time $t$ (seconds), in feet per second, of a runner. What does $I$ (defined above) represent?

$\int_1^7 v(t)dt$ gives the net change in position from $t=1$ to $t=7$.

(b) (4 points) Approximate $I$ with $R_2$, or explain why you cannot evaluate it with the information given.

$n=2 \rightarrow \Delta t = \frac{b-a}{n} = \frac{7-1}{2} = 3.$

$R_2 = v(4)\Delta t + v(7)\Delta t$

$= (0.6)3 + (-1.4)3$

$= \boxed{-2.4}$

(c) (4 points) Approximate $I$ with $M_3$, or explain why you cannot evaluate it with the information given.

$n=3 \rightarrow \Delta t = \frac{7-1}{3} = 2.$

$M_3 = v(2)\Delta t + v(4)\Delta t + v(6)\Delta t$

$= (2.9)2 + (0.6)2 + (-0.9)2 = \boxed{5.2}$

(d) (3 points) $v(t)$ appears to be decreasing and concave up. Assuming that it is, will $T_6$ be an underestimate for $I$, an overestimate for $I$, or can we not tell? (No explanation necessary.)

Overestimate because $v(t)$ is concave up.

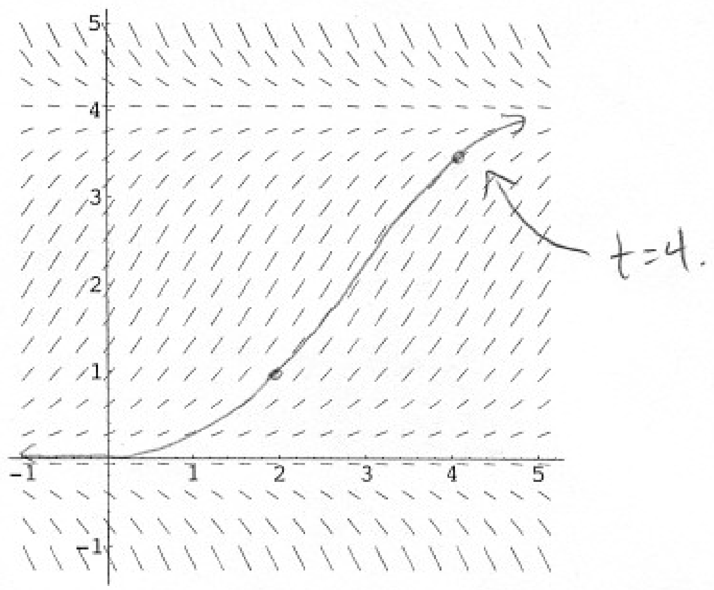



Figure 1: $y' = .4y(4-y)$

3. (5 points) Consider the differential equation $y' = .4y(4-y)$. The slope field is given above.

   (a) Sketch the solution for which $y(2) = 1$. (Be sure to sketch the solution in both directions, i.e., to the left and right of $t = 2$.) Label the point on your sketch where $t = 4$.

   (b) From your solution sketch, estimate the value of $y(4)$.

$y(4) \approx 3.5$.

4. (10 points) Consider the initial value problem

$$\begin{cases} y' = 2ty \\ y(1) = -1 \end{cases}$$

(This has nothing to do with the previous problem or its slope field!) Use Euler's method with two steps to estimate $y(3)$.

$\Delta t = 1.$

$t_0 = 1$

$y_0 = -1$

$m_0 = 2(1)(-1) = -2$

$\Delta y = m_0 \Delta t = -2 \cdot 1 = -2.$

$t_1 = 2$

$y_1 = y_0 + \Delta y = -1 - 2 = -3$

$m_1 = 2(2)(-3) = -12$

$\Delta y = m_1 \Delta t = -12 \cdot 1 = -12$

$t_2 = 3$

$y_2 = y_1 + \Delta y$

$= -3 - 12$

$= -15.$

$\boxed{y(3) \approx -15.}$

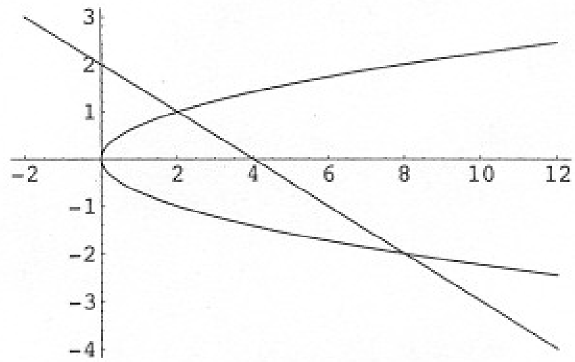



Figure 2: Graphs of $x = 2y^2$ and $y = -\frac{x}{2} + 2$

5. (10 points each) Consider the region in the $xy$-plane bounded by the graphs $x = 2y^2$ and $y = -\dfrac{x}{2} + 2$. In each part below, set up, but don't evaluate, an integral, or if necessary, a sum of two integrals, that represents the quantity given below.

   (a) The area of the region, where the integral(s) is (are) done with $dx$.

$x = 2y^2 \rightarrow y = \pm\sqrt{\frac{x}{2}}$.

$$A = \int_{x=0}^{x=2} \underbrace{\sqrt{\frac{x}{2}}}_{\text{top}} - \underbrace{\left(-\sqrt{\frac{x}{2}}\right)}_{\text{bottom}} dx + \int_{x=2}^{x=8} \underbrace{\left(-\frac{x}{2}+2\right)}_{\text{top}} - \underbrace{\left(-\frac{\sqrt{x}}{\sqrt{2}}\right)}_{\text{bottom}} dx.$$

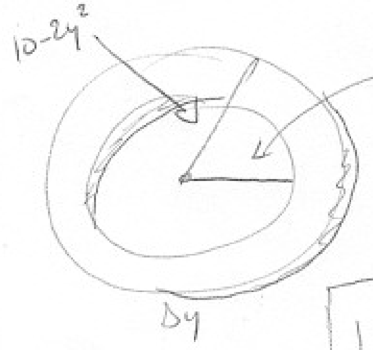
   (b) The area of the region, where the integral(s) is (are) done with $dy$.

$y = -\frac{x}{2} + 2 \rightarrow 2y = -x + 4 \rightarrow x = 4 - 2y$.

$$A = \int_{y=-2}^{y=1} \underbrace{(4-2y)}_{\text{right fn}} - \underbrace{(2y^2)}_{\text{left fn.}} dy$$

(Problem 5 continued. Set up the integral(s) to calculate the quanitity. You do not need to evaluate the integral(s).)

(c) The volume of the solid obtained by rotating the region around the line $x = 10$.




$R_{outer} = 10 - x = 10 - 2y^2$

$R_{inner} = 10 - x = 10 - (4-2y)$

$V_{slice} = \left(\pi\left(10-2y^2\right)^2 - \pi\left(10-(4-2y)\right)^2\right)\Delta y$

$$\boxed{V = \pi \int_{y=-2}^{y=1} (10-2y^2)^2 - (10-(4-2y))^2 \, dy}$$

6. (10 points) Let $I = \int_2^4 x \ln x \, dx$. Approximating $I$ with a midpoint sum $M_n$, how large must $n$ be to guarantee accuracy to within $\frac{1}{1000}$ of $I$?

$f(x) = x \ln x$

$f'(x) = \ln x + 1$

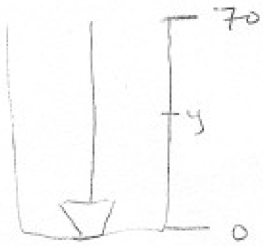
$f''(x) = \frac{1}{x}$. $\quad |f''(x)| \leq \frac{1}{2}$ on $[2,4]$. $\rightarrow K_2 = \frac{1}{2}$.

$|I - M_n| \leq \dfrac{\frac{1}{2}(4-2)^3}{24n^2} \leq \dfrac{1}{1000}$, so $\dfrac{1}{6n^2} \leq \dfrac{1}{1000}$

So $n^2 \geq \frac{1000}{6} = \frac{500}{3} = 166.\bar{6}$.

$n \geq 12.9\ldots \rightarrow$ $\boxed{n = 13 \text{ works.}}$

7. (10 points) A 40 pound bucket, at the bottom of a 70 foot well, is connected to the top by rope that weighs 2 pounds per foot. How much work is required to raise the bucket (and rope) to the top of the well? (Set up the integral - you do not need to evaluate it.)




When the bucket is $y$ feet above the bottom of the well, the total force of the bucket and rope is

$$F(y) = \underbrace{40}_{\text{bucket}} + \underbrace{(70-y)}_{\text{length of rope}} \cdot \underbrace{2}_{\text{density of rope}}.$$

Total work to move from $y=0$ to $y=70$ is

$$W = \int_0^{70} F(y)\,dy = \boxed{\int_0^{70} 40 + (70-y)\cdot 2 \, dy.}$$

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | TOTAL |
|---|---|---|---|---|---|---|---|
| | | | | | | | |