

# MATH 106, Section D

## Test # 1 SOLUTIONS

1. (20 points) Evaluate the integral.  $\int \frac{\cos(\ln x)}{x} dx$

$$\boxed{u = \ln x \implies du = \frac{1}{x} dx} \implies \int \frac{\cos(\ln x)}{x} dx = \int \cos u du = \sin u + C = \boxed{\sin \ln x + C}.$$

2. (20 points) Find the minimal value of  $n$  such that  $R_n$  approximates the value of the integral with the absolute error

$$\text{less than or equal to } 0.007. \int_1^e x \ln x dx$$

First of all, we want to find  $K_1$ , to do this we have to find critical points of  $f'(x)$  and compare the values of  $f'$  at them with the values of  $f'$  at the end-points of the interval. Among all these values we need the one with the largest absolute value, this absolute value is  $K_1$ , i.e., the minimal number that bounds  $|f'(x)|$  from above. Hence,

$$f(x) = x \ln x \implies f'(x) = \ln x + 1 \implies f''(x) = \frac{1}{x} \neq 0 \implies \text{no critical points.}$$

Now we have the following values:  $f'(1) = 1$ ,  $f'(e) = 2$ . Thus  $K_1 = 2$ , therefore, the bound gives us

$$0.007 \leq \frac{K_1 (b-a)^2}{2n} = \frac{2(e-1)^2}{2n} \implies n \leq \frac{(e-1)^2}{0.007} \approx 421.7846 \implies \boxed{n = 422}.$$

3. (20 points) Find the area of the region in the  $xy$ -plane bounded by the curves  $y = e^{2x}$ ,  $y = e^x$ , and  $x = \ln 2$ .

$$\begin{aligned} A &= \int_0^{\ln 2} (e^{2x} - e^x) dx = \int_0^{\ln 2} e^{2x} dx - \int_0^{\ln 2} e^x dx = \frac{1}{2} e^{2x} \Big|_0^{\ln 2} - e^x \Big|_0^{\ln 2} = \frac{1}{2} (e^{2 \ln 2} - e^0) - (e^{\ln 2} - e^0) \\ &= \frac{1}{2} ((e^{\ln 2})^2 - 1) - (2 - 1) = \frac{1}{2} (4 - 1) - 1 = \frac{3}{2} - 1 = \boxed{\frac{1}{2}}. \end{aligned}$$

4. (20 points) Find the length of the curve.  $y = \ln x - \frac{x^2}{8}$ ,  $x \in [1, 4]$

$$\begin{aligned} L &= \int_1^4 \sqrt{1 + \left( (\ln x - \frac{x^2}{8})' \right)^2} dx = \int_1^4 \sqrt{1 + \left( \frac{1}{x} - \frac{x}{4} \right)^2} dx = \int_1^4 \sqrt{1 + \left( \frac{1}{x} \right)^2 - 2 \cdot \frac{1}{x} \cdot \frac{x}{4} + \left( \frac{x}{4} \right)^2} dx \\ &= \int_1^4 \sqrt{1 + \left( \frac{1}{x} \right)^2 - \frac{1}{2} + \left( \frac{x}{4} \right)^2} dx = \int_1^4 \sqrt{\left( \frac{1}{x} \right)^2 + \frac{1}{2} + \left( \frac{x}{4} \right)^2} dx = \int_1^4 \sqrt{\left( \frac{1}{x} + \frac{x}{4} \right)^2} dx = \int_1^4 \left| \frac{1}{x} + \frac{x}{4} \right| dx \\ &= \int_1^4 \left( \frac{1}{x} + \frac{x}{4} \right) dx = \int_1^4 \frac{1}{x} dx + \frac{1}{4} \int_1^4 x dx = \ln |x| \Big|_1^4 + \frac{1}{8} x^2 \Big|_1^4 = \ln 4 - \ln 1 + \frac{1}{8} (16 - 1) = \boxed{2 \ln 2 + \frac{15}{8}}. \end{aligned}$$

5. (20 points) Find the volume of the solid of revolution formed when the region bounded by  $y = 1 - \sqrt{x}$ ,  $y = x + 1$ , and  $x = 1$  is revolved about the line  $y = 3$ .

$$\begin{aligned} \text{The area of the cross section: } A(x) &= \pi (3 - (1 - \sqrt{x}))^2 - \pi (3 - (x + 1))^2 = \pi (2 + \sqrt{x})^2 - \pi (2 - x)^2 \\ &= \pi (4 + 4\sqrt{x} + x - (4 - 4x + x^2)) = \pi (4\sqrt{x} + 5x - x^2). \end{aligned}$$

$$\begin{aligned} V &= \int_0^1 A(x) dx = \pi \int_0^1 (4\sqrt{x} + 5x - x^2) dx = \pi \left[ \frac{4}{3/2} x^{3/2} \Big|_0^1 + \frac{5}{2} x^2 \Big|_0^1 - \frac{1}{3} x^3 \Big|_0^1 \right] \\ &= \pi \left[ \frac{8}{3} + \frac{5}{2} - \frac{1}{3} \right] = \pi \frac{16+15-2}{6} = \pi \frac{29}{6} = \boxed{\frac{29\pi}{6}}. \end{aligned}$$

**6. (20 points)** A cylindrical gasoline tank with vertical axis of symmetry, radius 20 feet, and height 60 feet is located on the ground. One third of the tank is empty. Gasoline in the tank weighs  $43 \text{ lb/ft}^3$ . Find the amount of work needed to pump all the gasoline in the tank to a nozzle that is 15 feet above the top of the tank.

$$W = \int_a^b \rho g A(y) (h - y) dy \text{ where } a = 0, b = \frac{2}{3} \cdot 60 = 40, \rho g = 43, A(y) = \pi r^2 = 400\pi, \text{ and } h = 60 + 15 = 75.$$

$$\begin{aligned} W &= \int_0^{40} 43 \cdot 400\pi (75 - y) dy = 17200\pi \left[ 75 \int_0^{40} dy - \int_0^{40} y dy \right] = 17200\pi \left[ 75y \Big|_0^{40} - \frac{1}{2}y^2 \Big|_0^{40} \right] \\ &= 17200\pi \left[ 3000 - \frac{1}{2} \cdot 1600 \right] = 37,840,000\pi \approx \boxed{118,877,866 \text{ ft-lb}}. \end{aligned}$$