Math 206 — First Midterm

February 1, 2012

Name: EXAM SOLUTIONS

Instructor: ______________________ Section: __________

1. **Do not open this exam until you are told to do so.**
2. This exam has 8 pages including this cover AND IS DOUBLE SIDED. There are 8 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions.
5. Show an appropriate amount of work (including appropriate explanation). Include units in your answer where that is appropriate. Time is of course a consideration, but do not provide no work except when specified.
6. You may use any previously permitted calculator. However, you must state when you use it.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones and hats.
9. Remember that this is a chance to show what you've learned, and that the questions are just prompts.

Problem	Points	Score
1	15	
2	20	
3	14	
4	14	
5	12	
6	09	
7	14	
8	02	
Total	100	

1. [15 points] Match the following three curves and surfaces with the equations (there are three unused equations). No partial credit, no explanation needed (5 points for each correct match...).

1.$x^2 + 2x - y^2 = (z+2)^2$

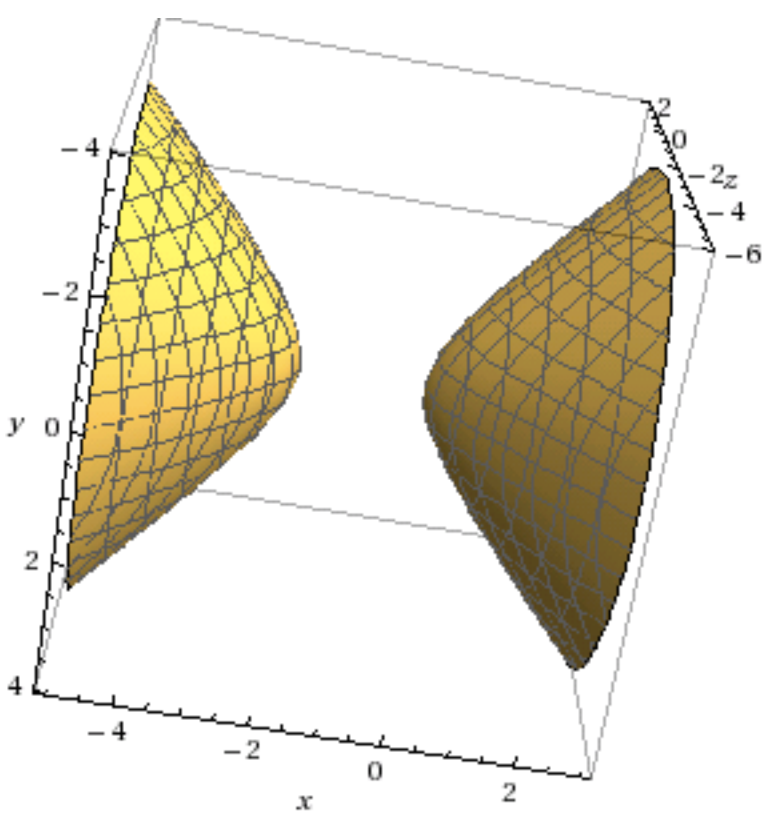
2.$[t(t-1), t^2(t-1), t(t-1)^2]$ for any real t

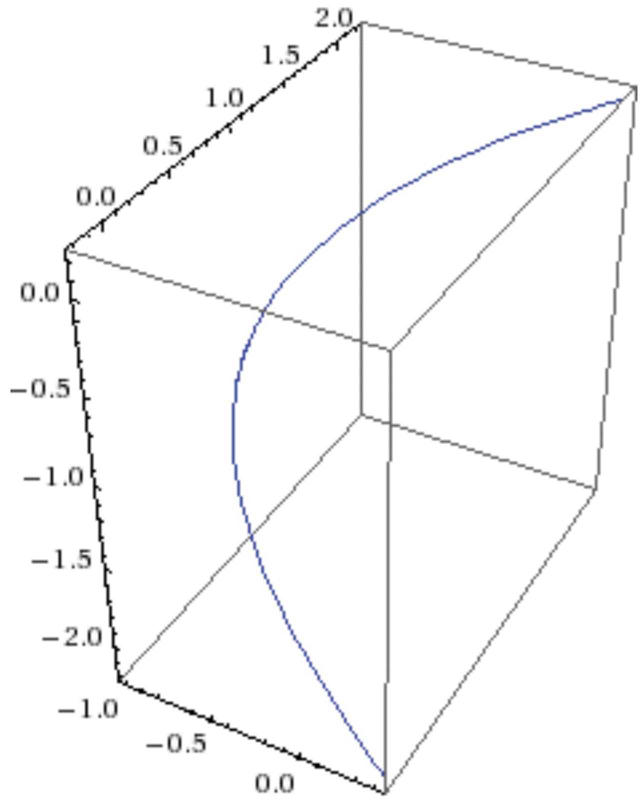
3.$[3\sin(2t), 2\cos(3t), t]$ for any real t

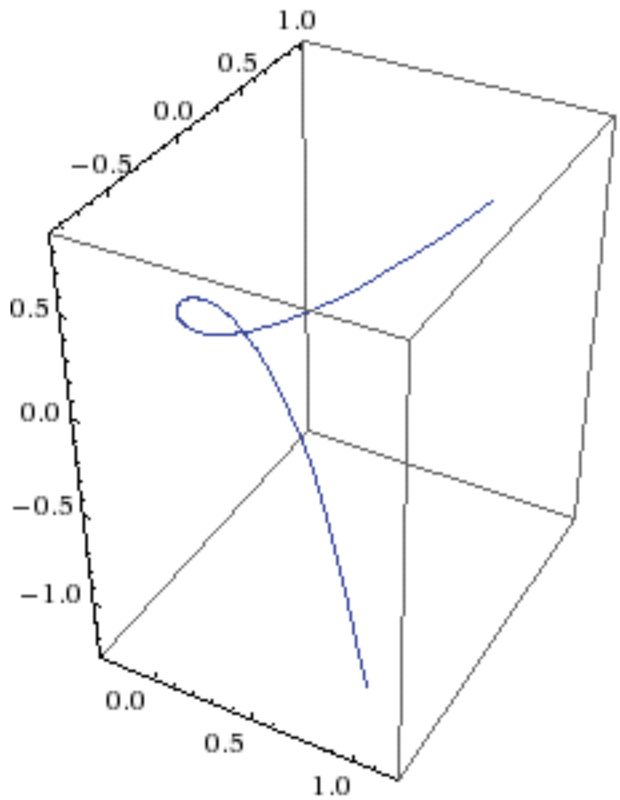
4.$x = e^{y^2+z^2}$

5.$z = e^{x^2+y^2}$

6.$[t^2-1, t+1, t-1]$ for any real t

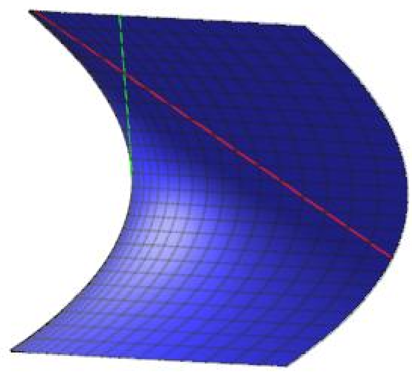






Solution: 1, 6, 2.

2. [20 points]
The surface described by $z = x^2 - y^2$ is the hyperbolic paraboloid (or "saddle") that we've seen in class (or in the image below). What you may not know, is that this is also the general shape used for cooling towers in power plants (see below). The reason is that even though this is a curved surface, there are straight lines for steel beams contained inside it (note the red and green lines). Let's explore:



a. [8 points] Check that the two functions below parametrize two straight lines which are contained in the surface $z = x^2 - y^2$? Briefly explain your work (including whether these ARE lines).

- $f(t) = [t+1, t-1, 4t]$ for any real t
- $g(s) = [s+1, 1-s, 4s]$ for any real t

Solution: These are both lines because every function in them is a linear function (no exponents). To test whether these are in the surface, we plug the functions into $z = x^2 - y^2$ and get:

$$(t+1)^2 - (t-1)^2 = 4t$$

and

$$(s+1)^2 - (1-s)^2 = 4s.$$

b. [5 points] Find the point where these two lines intersect?

Solution: We need that

$$\begin{aligned} t+1 &= s+1 \\ t-1 &= 1-s \\ 4t &= 4s. \end{aligned}$$

So we see that $t = s$ and $t = 1$. This gives us the point $[2, 0, 4]$.

c. [7 points] Find the tangent plane of the hyperbolic paraboloid at that point.

Solution: We calculate the Jacobian of $z = x^2 - y^2$ and get $[2x \;\; 2y]$. Plugging in $(2, 0)$ we get $[4 \;\; 0]$. Also $f(2, 0) = 4 - 0 = 4$. So we get
$z = 4 + [4 \;\; 0][x - 2 \;\; y] = 4 + 4x - 8 = 4x - 4$.

3. [14 points] Calculate the following limits, or demonstrate that they do not exist:

a. [7 points]

$$f(x,y) = \frac{x^3 y}{x^6 + y^2}$$

What is $\lim_{(x,y)\to(0,0)} f(x,y)$ or does it not exist?

Solution: This limit does not exist. If we "walk" down the line $x = 0$ we get the limit is zero. If we walk down the line $y = x^3$ we get the limit is $1/2$, and since they do not agree, the limit does not exist.

b. [7 points] Does

$$\lim_{t\to 0} \frac{(x+t)^2 \cos(y) - x^2 \cos(y)}{t}$$

exist? If so what is it?

Solution: This is the definition of the partial derivative of $x^2 \cos(y)$ with respect to x. So the limit is $2x \cos(y)$.

4. [14 points] Explain why two level curves of a function $f(x, y)$ cannot intersect.

Solution: A level curve is the set of points (x, y) which satisfy $f(x, y) = c$ for a fixed number c. If we have two different level curves (two different constants) then they can only intersect at a point (a, b) if $f(a, b) = c$ and $f(a, b) = d$. Since f is a function, it cannot have two different outputs for the same input.

5. [12 points]

a. [6 points] Let $u = (-3, 2, 3)$. Find a vector v of length 1 so that $u + v$ is as long as possible.

Solution: To make the size of $u + v$ as large as possible, we want v continuing in the same direction as u, but with size 1. So we take the normalization of u: $u/|u|$. This gives $(-3/\sqrt{22}, 2/\sqrt{22}, 3/\sqrt{22})$.

b. [6 points] Let $u = (-3, 2, 3)$. Find a vector w of length 1 so that $|u \times w|$ is as small as possible.

Solution: To get a cross-product with norm zero, we need u and w to be parallel. This means we can just choose the same w as we had for v in the previous part.

6. [9 points] Sketch the image of the square ($1 \le x \le \mathrm{e}$ and $0 \le y \le 1$) under the function $f(x,y) = (\ln(x)y, x\,\mathrm{e}^y)$ We put the borders of the square into the function and get:

$$\begin{aligned} f(t,0) &\mapsto (0,t) \quad \text{for} \quad 1 \le t \le \mathrm{e} \\ f(\mathrm{e},t) &\mapsto (t, \mathrm{e}^{t+1}) \quad \text{for} \quad 0 \le t \le 1 \\ f(t,1) &\mapsto (\ln(t), \mathrm{e}\,t) \quad \text{for} \quad 1 \le t \le \mathrm{e} \\ f(1,t) &\mapsto (0, \mathrm{e}^t) \quad \text{for} \quad 0 \le t \le 1 \end{aligned}$$

Putting this all together the first is a line starting at $(0,1)$ and ending at $(0,\mathrm{e})$. The second is the curve $y = \mathrm{e}^{x+1}$ starting at $(0,\mathrm{e})$ and ending at $(1,\mathrm{e}^2)$. The third is the same curve as the second, but in reverse. And the fourth is the same line as the first but in reverse. So we just get a line and a curve as the entire image.

7. [14 points]

a. [7 points] If $f(t) = (t^3 - t, t^2 + 1, t - 5)$ what is $f'(2)$?

Solution: We differentiate each coordinate in turn and get $f'(t) = (3t^2 - 1, 2t, 1)$. We plug in 2 and get $f'(2) = (11, 4, 1)$.

b. [7 points] What is the tangent line at $f(2)$ to the curve parametrized in the first part of this problem?

Solution: It is the line passing through the point $f(2)$ which has slope determined by $(11, 4, 1)$. We can calculate that $f(2) = (6, 5, -3)$ and so the line is $(6, 5, -3) + (t - 2)(11, 4, 1)$.

8. [2 points] Who do you think will win the superbowl? The Ravens or the Forty-niners?

Solution: The Ravens won a nail-biter.