MATH106A,B CALCULUS II - PROF. P. WONG

EXAM I - FEBRUARY 1, 2008

NAME:

Instruction: Read each question carefully. Explain **ALL** your work and give reasons to support your answers.

Advice: DON'T spend too much time on a single problem.

Problems	**Maximum Score**	**Your Score**
1.	20	
2.	20	
3.	20	
4.	20	
5.	20	
Total	100	

1.(10 pts.)(a) Find the **exact value** (by the Fundamental Theorem of Calculus) of the definite integral

$$\int_1^e \frac{1+(\ln x)^2}{x}\ dx.$$

Let $u = \ln x$**. Then** $du = \frac{1}{x}dx$**. When** $x = 1, u = 0$ **and when** $x = e$**,** $u = 1$**. Thus,**

$$\int_1^e \frac{1+(\ln x)^2}{x}\ dx = \int_0^1 1 + u^2\ du$$
$$= u + \frac{u^3}{3}\Big|_0^1 = \frac{4}{3}.$$

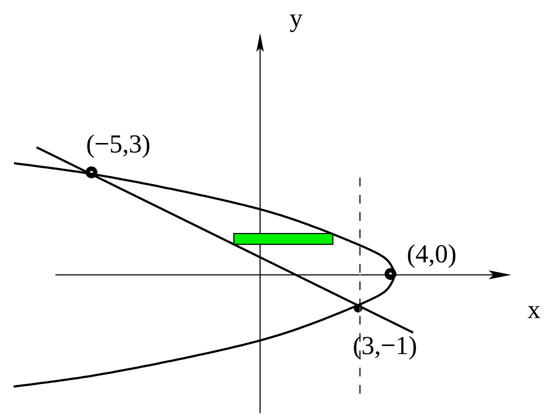
(10 pts.)(b) Evaluate the indefinite integral

$$\int \frac{x}{\sqrt{1-x^4}}\ dx.$$

Let $u = x^2$**. Then** $du = 2x\ dx$ **or** $x\ dx = \frac{1}{2}dx$**. Thus,**

$$\int \frac{x}{\sqrt{1-x^4}}\ dx = \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}}$$
$$= \frac{1}{2}\arcsin u + C = \frac{1}{2}\arcsin(x^2) + C.$$

2.(20 pts.) Find the area of the region bounded by the curve $x = 4 - y^2$ and the line $2y = 1 - x$. [Hint: sketch a picture of the region by determining the points of intersections between the curve and the line]



When the curve $x = 4 - y^2$ and the line $2y = 1 - x$ meet, we have $x = 4 - y^2 = 1 - 2y$ or $y^2 - 2y - 3 = (y-3)(y+1) = 0$ so $y = -1$ or $y = 3$. The points of intersections are $(-5, 3)$ and $(3, -1)$.

We will use horizontal slices to find the area. [Note that if one uses vertical slices, there will be a problem in expressing the area as one single integral because the lower and upper bounds for the vertical slices between $x = 3$ and $x = 4$ are from the same parabola.] The area of the bounded region is given by

$$
\begin{aligned}
A &= \int_{-1}^{3} [4 - y^2] - [1 - 2y] \ dy \\
&= \int_{-1}^{3} 3 - y^2 + 2y \ dy \\
&= 3y - \frac{y^3}{3} + y^2 \Big|_{-1}^{3} \\
&= (9 - 9 + 9) - (-3 + \frac{1}{3} + 1) = \frac{32}{3}.
\end{aligned}
$$

Here, the x-coordinate of the right end point of a typical slice (see figure) is given by $(4 - y^2 0$ while that of the left end point is given by $(1 - 2y)$.

If you use vertical slices, $A = \int_{-5}^{3} \sqrt{4-x} - \frac{1-x}{2} \ dx + \int_{-1}^{1} (4 - y^2) - 3 \ dy$. *The second integral is with respect to y (using horizontal slices).*

3. (10 pts.)(a) Consider a function h given by the following table.

x	1	1.5	2	2.5	3	3.5	4
$h(x)$	-1	2	1	0	-2	3	1

Find R_6, M_3 using the right-hand sum and the mid-point rule respectively for estimating the definite integral $\int_1^4 h(x)\ dx$.

$$R_6 = [h(1.5) + h(2) + h(2.5) + h(3) + h(3.5) + h(4)] \cdot \Delta x$$
$$= [2 + 1 + 0 + (-2) + 3 + 1] \cdot (0.5) = 2.5$$

and

$$M_3 = [h(1.5) + h(2.5) + h(3.5)] \cdot \Delta x \qquad \textbf{(here the } \Delta x \textbf{ is 1)}$$
$$= [2 + 0 + 3] \cdot (1) = 5$$

(10 pts.)(b) Recall that the error committed by using the left hand sum approximation L_n is less than or equal to $\frac{K_1 \cdot (b-a)^2}{2n}$ where $|f'(x)| \leq K_1$ for some constant K_1 over the interval $[a, b]$. Use this result to give an upper bound for the error committed by L_{10} for

$$I = \int_0^2 (\sin x) e^x\ dx.$$

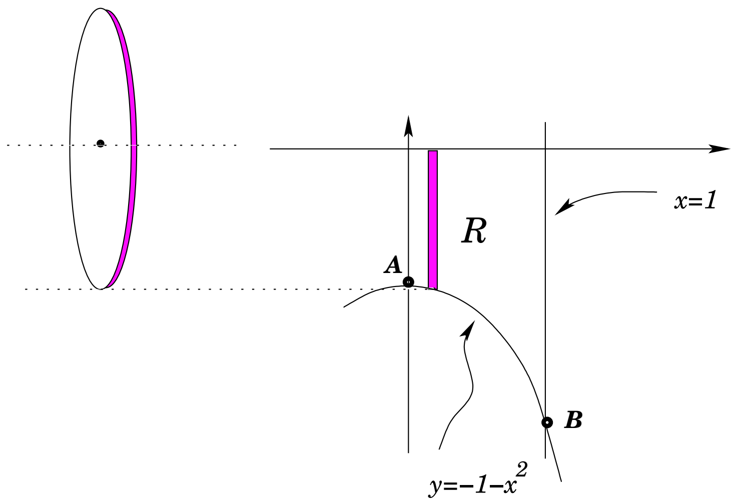
Here, $f(x) = \sin x\ e^x$ **so** $f'(x) = \cos x\ e^x + \sin x\ e^x = e^x(\sin x + \cos x)$**. Now,** $|f'(x)| = e^x |\sin x + \cos x| \leq e^x(1 + 1)$**. Over the interval** $[0, 2]$**,** $|f'(x)| \leq 2e^2$ **so we can choose** $K_1 = 2e^2$**. It follows that**

$$|I - L_{10}| \leq \frac{2e^2(2-0)^2}{2(10)} = \frac{2e^2}{5}.$$

One can also use the calculator to estimate that $|f'(x)| < 5$ **(note that the maximum of** $f'(x)$ **does NOT occur at** $x = 2$**) so that one can choose** K_1 **to be** 5**.**

4. Let R be the region bounded by the curve $y = -1 - x^2$, the line $x = 1$, the x-axis and the y-axis.

(15 pts.) Find the **exact volume** of the solid obtained from rotating the region R around the x-axis.



A typical slice of this solid is a circular disk with thickness Δx so that the integral for the volume is with respect to x. The volume is given by

$$V = \int_0^1 \pi(-1-x^2)^2 \; dx = \pi \int_0^1 1 + 2x^2 + x^4 \; dx = \pi \left(x + \frac{2x^3}{3} + \frac{x^5}{5} \right)\Big|_0^1$$
$$= \pi \left(1 + \frac{2}{3} + \frac{1}{5} \right) = \frac{28\pi}{15}.$$

(5 pts.) SET UP (**do not evaluate**) a definite integral representing the arc length of the portion of the curve from A to B.

The point A has coordinates $(0, -1)$ and B has coordinates $(1, -2)$. The arc length of the path from A to B is given by

$$L = \int_0^1 \sqrt{1 + [f'(x)]^2} \; dx = \int_0^1 \sqrt{1 + [-2x]^2} \; dx \qquad \textbf{(where } f(x) = -1 - x^2\textbf{)}$$
$$= \int_0^1 \sqrt{1 + 4x^2} \; dx.$$

5. (10 pts.)(a) Consider the initial value problem

$$\frac{dy}{dx} = \frac{x}{x+y}$$

with $y(1) = 1$.

Use Euler's method to estimate the value $y(2)$ (when $x = 2$) of the solution using two steps with initial point $(1, 1)$. DO THIS BY HAND and show all your steps.

The initial y-value (step 0) is $y_0 = 1$ when $x_0 = 1$. Since we use two steps from $x = 1$ to $x = 2$, the step size is $\Delta x = 0.5$. Now, the next approximation according to Euler's method is $y_1 = y_0 + f(x_0, y_0) \cdot \Delta x$ where $f(x, y) = \frac{dy}{dx}$. Thus, $y_1 = 1 + \frac{1}{1+1} \cdot (0.5) = \frac{5}{4} = 1.25$. Next, $y_2 = y_1 + f(x_1, y_1) \cdot \Delta x$ where $x_1 = 1.5$. Thus, $y_2 = 1.25 + \frac{1.5}{1.5+1.25} \cdot (0.5) = \frac{67}{44}$. In other words, $y(2) \approx \frac{67}{44}$.

(10 pts.)(b) Use the technique of separation of variables to solve the following Initial Value Problem

$$y' = \frac{y}{\sqrt{1-x^2}}, \qquad y(0) = 1.$$

Since $\frac{dy}{dx} = y' = \frac{y}{\sqrt{1-x^2}}$, it follows, by separating the variables, that

$$\int \frac{1}{y}\, dy = \int \frac{dx}{\sqrt{1-x^2}}.$$

Integrating both sides, we have

$$\ln |y| = \arcsin x + C.$$

Since $y = 1$ when $x = 0$, $0 = \ln 1 = \arcsin 0 + C$ which implies that $C = 0$. Solving for y, we obtain

$$y = e^{\arcsin x}.$$