

NAME: SOLUTIONS

Math 106 - Exam 1 - February 1, 2008

Instructions: Show all of your work and circle your final answers. Your work should flow logically and be easy to follow. Cross out any work that you do not want considered. Calculators are allowed, but notes and books are not.

Problem	Max	Score
1	18	
2	10	
3	16	
4	6	
5	10	
6	10	
TOTAL	70	

1. (18 points) Evaluate the following:

(a)  $\int_1^e \frac{\cos(\ln x)}{x} dx.$

Let  $u = \ln x$ , so  $du = \frac{1}{x} dx$ .  $x=1 \rightarrow u = \ln x = \ln 1 = 0$

$x=e \rightarrow u = \ln e = 1$

Then  $\int_{x=1}^{x=e} \frac{\cos(\ln x)}{x} dx = \int_{u=0}^{u=1} \cos u du$

$= \sin u \Big|_0^1 = \sin(1) - \sin(0)$

$= \sin(1) - 0$

$\boxed{= \sin(1)}$

(b)  $\int x\sqrt{2x-1} dx.$

Let  $u = 2x-1$ , so  $du = 2dx$ . (So  $\frac{du}{2} = dx$ .)

$\rightarrow = \int x\sqrt{u} \cdot \frac{du}{2}$   $x$  in terms of  $u$ ?  $u = 2x-1,$

So  $\frac{u+1}{2} = x.$

$= \int \left(\frac{u+1}{2}\right) \cdot \sqrt{u} \cdot \frac{du}{2}$

$= \frac{1}{4} \int (u+1)\sqrt{u} du$

$= \frac{1}{4} \int u^{3/2} + u^{1/2} du$

$= \frac{1}{4} \left( \frac{2}{5} u^{5/2} + \frac{2}{3} u^{3/2} + C \right)$

$= \frac{2}{20} (2x-1)^{5/2} + \frac{2}{12} (2x-1)^{3/2} + C$

$\boxed{= \frac{1}{10} (2x-1)^{5/2} + \frac{1}{6} (2x-1)^{3/2} + C}$

2. (10 points) Let  $I = \int_{\frac{1}{2}}^{\frac{3}{2}} \ln x dx$ .

(a) How large must  $n$  be to ensure that  $M_n$  approximates  $I$  with error at most .001?

$M_n$  needs  $K_2$ , so we need  $f''(x)$ .

$$f(x) = \ln x, \quad f'(x) = \frac{1}{x}, \quad f''(x) = -x^{-2} = -\frac{1}{x^2}$$

On interval  $[\frac{1}{2}, \frac{3}{2}]$ , max value of  $|f''(x)|$  is 4, which occurs when  $x = \frac{1}{2}$ . ( $f''(x) = -\frac{1}{x^2}$  is negative and increasing, so its most negative value occurs at the start of the interval, which is  $x = \frac{1}{2}$ .) So, let  $K_2 = 4$ .

$$|I - M_n| \leq \frac{K_2 (b-a)^3}{24n^2} \leq \frac{1}{1000}$$

$$\frac{4(\frac{3}{2} - \frac{1}{2})^3}{24n^2} \leq \frac{1}{1000}$$

$$\sqrt{\frac{1000}{6}} \leq \sqrt{n^2}$$

$$12.9... \leq n$$

So, if  $n = 13$ , we have the desired accuracy.

(b) Will  $L_{100}$  underestimate or overestimate  $I$ ? Explain. (You do not need to evaluate  $L_{100}$ .)

$f(x) = \ln x$  is increasing on  $[\frac{1}{2}, \frac{3}{2}]$ . (See this with a graph or by noting its derivative,  $\frac{1}{x}$ , is positive.)

Hence, a left endpoint sum will be an underestimate.

3. (16 points) Consider the IVP  $y' = \frac{y^2}{1+t^2}$ ,  $y(0) = 1$ .

(a) By hand, use Euler's method with step size  $\Delta t = 1$  to approximate  $y(2)$ .

$$t_0 = 0, y_0 = 1, y' = \frac{y^2}{1+t^2}, \Delta t = 1.$$

$$t_1 = t_0 + \Delta t = 0 + 1 = 1$$

$$\begin{aligned} y_1 &= y_0 + \Delta y = y_0 + \left( \frac{y_0^2}{1+t_0^2} \right) \Delta t \\ &= 1 + \left( \frac{1^2}{1+0^2} \right) \cdot 1 = 1 + 1 = 2. \end{aligned}$$

$$t_1 = 1, y_1 = 2$$

$$t_2 = t_1 + \Delta t = 1 + 1 = 2.$$

$$\begin{aligned} y_2 &= y_1 + \Delta y = y_1 + \left( \frac{y_1^2}{1+t_1^2} \right) \Delta t \\ &= 2 + \left( \frac{2^2}{1+1^2} \right) \Delta t = 2 + \frac{4}{2} \cdot 1 = 4. \end{aligned}$$

$$t_2 = 2, y_2 = 4. \Rightarrow \boxed{y(2) \approx 4}, \text{ by Euler's method.}$$

(b) Using the technique of separation of variables, find the solution to the above IVP.

$$\frac{dy}{dt} = \frac{y^2}{1+t^2}$$

$$\int \frac{dy}{y^2} = \int \frac{dt}{1+t^2}$$

$$-\frac{1}{y} = \arctan(t) + C.$$

$$y = \frac{-1}{\arctan(t) + C}$$

General Solution.  $\int$

Then,  $y(0) = 1$ , so

$$1 = \frac{-1}{\arctan(0) + C}$$

$$1 = \frac{-1}{0 + C}$$

$$C = -1.$$

$$\rightarrow \boxed{y = \frac{-1}{\arctan(t) - 1}}$$

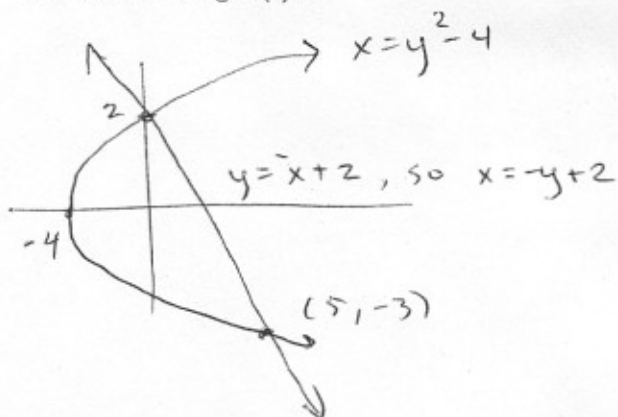
4. (6 points) Write down an integral that gives the length of the curve  $y = \sqrt{x}$  from  $x = 1$  to  $x = 4$ .

$$f(x) = \sqrt{x} \rightarrow f'(x) = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}}$$

$$f'(x)^2 = \left(\frac{1}{2\sqrt{x}}\right)^2 = \frac{1}{4x}$$

$$\text{Arc length} = \int_1^4 \sqrt{f'(x)^2 + 1} dx = \int_1^4 \sqrt{\frac{1}{4x} + 1} dx$$

5. (10 points) Consider the region in the  $xy$ -plane bounded by the graphs of  $x = y^2 - 4$  and  $y = -x + 2$ . Find an integral (or sum of integrals) that represents the area of the region. You do not need to evaluate the integral(s).



Intersection?

$$x = y^2 - 4$$

$$y = -x + 2$$

Sub in.

$$x = (-x + 2)^2 - 4$$

$$x = x^2 - 4x + 4 - 4$$

$$x = x^2 - 4x$$

$$0 = x^2 - 5x$$

$$0 = x(x - 5)$$

$$x = 0 \text{ or } x = 5$$

↓

$$y = -0 + 2 = 2$$

↓

$$y = -5 + 2 = -3$$

pts (0, 2), (5, -3)

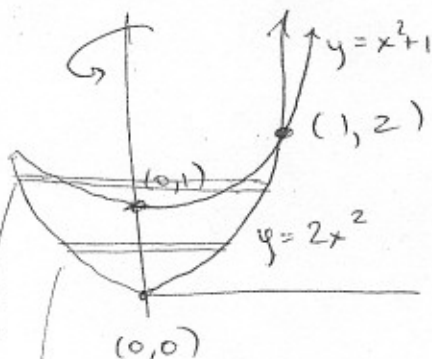
$$\text{Area} = \int_{y=-3}^{y=2} (-y+2) - (y^2-4) dy$$

Or... in terms of  $x$ , top half of parabola is  $y = \sqrt{x+4}$ , bottom half is  $y = -\sqrt{x+4}$ .

$$A = \int_{x=-4}^{x=0} \sqrt{x+4} - (-\sqrt{x+4}) dx + \int_{x=0}^{x=5} (-x+2) - (-\sqrt{x+4}) dx$$

6. (10 points) Consider the region in the first quadrant of the  $xy$ -plane that is bounded by the graphs of  $y = x^2 + 1$ ,  $y = 2x^2$ , and  $x = 0$ . Write down an integral (or a sum of integrals) that represents the volume of the solid obtained by revolving this region around the  $y$ -axis. You do not need to evaluate the integral(s).

Intersection:  $x^2 + 1 = 2x^2 \rightarrow x^2 = 1 \rightarrow x = \pm 1$ . Take  $x = +1$ .



Horizontal slices. For  $y$  in  $[0, 1]$ , cross section is a disc. For  $y$  in  $[1, 2]$ , have washers. Do them separately.



Radius =  $x$ .

$V_{disc} = \pi (x)^2 dy$ . On the graph  $y = 2x^2$ ,  $\frac{y}{2} = x^2$ .

So  $V_{disc} = \pi \left(\frac{y}{2}\right) dy$ .

Volume from  $y=0$  to  $y=1$  is  $\int_{y=0}^{y=1} \pi \frac{y}{2} dy$ .



Outer radius is  $x$ , which on the graph  $y = 2x^2$  is  $x = \sqrt{\frac{y}{2}}$ .

Inner radius is  $x$ , which on the graph  $y = x^2 + 1$  is  $x = \sqrt{y-1}$ .

$$A_{washer} = \pi \left(\sqrt{\frac{y}{2}}\right)^2 - \pi (\sqrt{y-1})^2$$

$$= \pi \left(\frac{y}{2}\right) - \pi (y-1).$$

$$V_{washer} = \left(\pi \frac{y}{2} - \pi (y-1)\right) dy.$$

Volume from  $y=1$  to  $y=2$  is  $\int_{y=1}^{y=2} \pi \frac{y}{2} - \pi (y-1) dy$ .

Total volume =  $\pi \int_0^1 \frac{y}{2} dy + \pi \int_1^2 \frac{y}{2} - (y-1) dy$ .