MATH 205A • Exam 1

Feb. 1, 2008

Name

Suggested solutions

Do ~~NOT~~ write here

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2	
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1. Read all the questions CAREFULLY
2. Show all work in the space provided
3. indicate clearly your FINAL ANSWER
4. BE NEAT

GOOD LUCK!

Here is a fact you may find useful:

For problem 1,

$$\left[\begin{array}{ccccc|ccccc} 0&1&q&1&2&1&0&0&0&0\\ 2&3&r&7&5&0&1&0&0&0\\ 1&2&s&4&3&0&0&1&0&0\\ 3&4&t&10&8&0&0&0&1&0\\ 2&2&u&6&3&0&0&0&0&1 \end{array}\right] \text{ is row equivalent to } \begin{array}{c} \begin{array}{cccccccccc} x_1&x_2&x_3&x_4&x_5&b_1&b_2&b_3&b_4&b_5 \end{array} \\ \left[\begin{array}{ccccc|ccccc} 1&0&-3&2&0&0&0&-1&0&1\\ 0&1&4&1&0&0&0&7/4&-3/4&1/4\\ 0&0&0&0&1&0&0&-1/2&1/2&-1/2\\ 0&0&0&0&0&1&0&-3/4&-1/4&3/4\\ 0&0&0&0&0&0&1&-3/4&-1/4&-1/4 \end{array}\right] \end{array}$$

just a reminder: This line represents

$2x_1 + 2x_2 + ux_3 + 6x_4 + 3x_5 = b_5$,

and this line tells us that $0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 = 0b_1 + b_2 - \frac{3}{4}b_3 - \frac{1}{4}b_4 - \frac{1}{4}b_5$

(or, $0 = b_2 - \frac{3}{4}b_3 - \frac{1}{4}b_4 - \frac{1}{4}b_5$)

(the handwritten info here is on the suggested soln', but NOT on the exam-as-given!)

1. Let $A = \begin{bmatrix} 0 & 1 & q & 1 & 2 \\ 2 & 3 & r & 7 & 5 \\ 1 & 2 & s & 4 & 3 \\ 3 & 4 & t & 10 & 8 \\ 2 & 2 & u & 6 & 3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}$ be in $\mathbf{R}^5$, and $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_5$ be the column vectors of A.

Here q, r, s, t and u (the components of $\mathbf{c}_3$) are scalars whose values you will find in problem 2; you do not need their values in problem 1.

1A. Write out the system of equations represented by the matrix equation $A\mathbf{x} = \mathbf{b}$.

$$\begin{cases} 0x_1 + x_2 + qx_3 + x_4 + 2x_5 = b_1 \\ 2x_1 + 3x_2 + rx_3 + 7x_4 + 5x_5 = b_2 \\ x_1 + 2x_2 + sx_3 + 4x_4 + 3x_5 = b_3 \\ 3x_1 + 4x_2 + tx_3 + 10x_4 + 8x_5 = b_4 \\ 2x_1 + 2x_2 + ux_3 + 6x_4 + 3x_5 = b_5 \end{cases}$$

1B. Let $\mathcal{S} = \text{Span}\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_5\}$. Find any conditions that $b_1, b_2, \ldots, b_5$ must satisfy so that $\mathbf{b}$ is in $\mathcal{S}$ or state explicitly that there are no special conditions. Note the information on page 0 may be useful!

the "super" augmented matrix tells us there are NO inconsistencies in the above system

$$\Longleftrightarrow$$

$$0 = b_1 - \tfrac{3}{4}b_3 - \tfrac{1}{4}b_4 + \tfrac{3}{4}b_5$$

AND $$0 = b_2 - \tfrac{3}{4}b_3 - \tfrac{1}{4}b_4 - \tfrac{1}{4}b_5$$

1C. Suppose that a given $\mathbf{b}$ is indeed in the span of these column vectors. Write the solutions of $A\mathbf{x} = \mathbf{b}$ in the form $\mathbf{p} + \mathbf{v}_h$ where $A\mathbf{p} = \mathbf{b}$ and $A\mathbf{v}_h = \mathbf{0}$. (Note that $\mathbf{p}$ will be in terms of $b_1, b_2, \ldots, b_5$).

the augmented matrix also tells us that when $\vec{b}$ is in the span,

$x_1 = (-b_3 + b_5) + 3x_3 - 2x_4$

$x_2 = (\tfrac{7}{4}b_3 + -\tfrac{3}{4}b_4 + \tfrac{1}{4}b_5) - 4x_3 - x_4$

x_3 is free

x_4 is free

$x_5 = (-\tfrac{1}{2}b_3 + \tfrac{1}{2}b_4 - \tfrac{1}{2}b_5)$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -b_3 + b_5 \\ \tfrac{7}{4}b_3 - \tfrac{3}{4}b_4 + \tfrac{1}{4}b_5 \\ 0 \\ 0 \\ -\tfrac{1}{2}b_3 + \tfrac{1}{2}b_4 - \tfrac{1}{2}b_5 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ -4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

1D. (Vocabulary) What are $\mathbf{p}$ and $\mathbf{v}_h$ called?

$\vec{p}$ is a particular soln of $A\vec{x} = \vec{b}$

$\vec{v}_h$ represents all soln's of the (corresponding) homogeneous eqn $A\vec{x} = \vec{0}$

problem 1, continued.

does $0 \overset{?}{=} 8 - \tfrac{3}{4}20 - \tfrac{1}{4}20 + \tfrac{3}{4}16 \overset{?}{=} 8-15-5+12 \overset{?}{=} 0$ ✓

& $0 = 24 - \tfrac{3}{4}20 - \tfrac{1}{4}20 - \tfrac{1}{4}16 = 24-15-5-4 \overset{?}{=} 0$ ✓

1E. Show whether the entries of $\mathbf{w} = \begin{bmatrix} 8 \\ 24 \\ 20 \\ 20 \\ 16 \end{bmatrix}$ satisfy all the conditions you found in 1B.

1F. Use (1C) to explicitly solve $A\mathbf{x} = \mathbf{w}$ (use the solution where any free variables are set to 0).

take $x_3 = x_4 = 0$ to get

$$\vec{x} = \begin{bmatrix} -b_3 + b_5 \\ \tfrac{7}{4} b_3 - \tfrac{3}{4} b_4 + \tfrac{1}{4} b_5 \\ 0 \\ 0 \\ -\tfrac{1}{2} b_3 + \tfrac{1}{2} b_4 - \tfrac{1}{2} b_5 \end{bmatrix} = \begin{bmatrix} -20 + 16 \\ \tfrac{7}{4}\cdot 20 - \tfrac{3}{4}\cdot 20 + \tfrac{1}{4}\cdot 16 \\ 0 \\ 0 \\ -\tfrac{1}{2}\cdot 20 + \tfrac{1}{2}\cdot 20 - \tfrac{1}{2}\cdot 16 \end{bmatrix} = \begin{bmatrix} -4 \\ 24 \\ 0 \\ 0 \\ -8 \end{bmatrix}$$

2. Let A be the same matrix as in problem 1, and again let $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_5$ be its column vectors.

2A. Can $\mathbf{c}_5$ be expressed as a linear combination of the other four columns? (Hint: is there a solution of $A\mathbf{x} = \mathbf{0}$ in which $\mathbf{c}_5$ appears with a nonzero weight?) Explain.

NO. otherwise we'd have $\alpha_1 \vec{c}_1 + \alpha_2 \vec{c}_2 + \alpha_3 \vec{c}_3 + \alpha_4 \vec{c}_4 - 1\cdot \vec{c}_5 = \vec{0}$, giving a soln of $A\vec{x} = \vec{0}$ in which the weight of $\vec{c}_5$ is non zero. BUT $\vec{v}_h$ on the previous page shows that in any soln of $A\vec{x} = \vec{0}$, $x_5 = 0$, and so we have a contradiction.

2B. Write the zero vector as a linear combination of the columns of A where $\mathbf{c}_3$ has a *nonzero* weight. Make sure your work shows how you found the weights.

Choosing $x_3 = 1$ and $x_4 = 0$ gives a soln of $A\vec{x} = \vec{0}$ in which the weight of column 3 is nonzero; we find $x_1 = 3$, $x_2 = -4$, $x_3 = 1$, $x_4 = 0$, $x_5 = 0$

so that $\boxed{3\vec{c}_1 - 4\vec{c}_2 + 1\vec{c}_3 + 0\vec{c}_4 + 0\vec{c}_5 = \vec{0}}$

2C. Use the answer to (2B) to write $\mathbf{c}_3$ as a linear combination of the other columns of A, and explicitly find the entries q, r, s, t and u of $\mathbf{c}_3$.

"solve" for $\vec{c}_3$ in the previous answer to obtain

$$\vec{c}_3 = -3\vec{c}_1 + 4\vec{c}_2 = -3\begin{bmatrix} 0 \\ 2 \\ 1 \\ 3 \\ 2 \end{bmatrix} + 4\begin{bmatrix} 1 \\ 3 \\ 2 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -6 \\ -3 \\ -9 \\ -6 \end{bmatrix} + \begin{bmatrix} 4 \\ 12 \\ 8 \\ 16 \\ 8 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 5 \\ 7 \\ 2 \end{bmatrix}$$

3. Let C= $\begin{bmatrix} 0 & 1 & 2 \\ 2 & 3 & 5 \\ 1 & 2 & 3 \end{bmatrix}$

3A. Find the RREF of C *by hand* and label all row operations. You can check your answer by calculator.

There are many different sequences of row ops that all lead to the same RREF.

Here's just one: (changed rows are in **BOLD**)

$$\begin{bmatrix} 0 & 1 & 2 \\ 2 & 3 & 5 \\ 1 & 2 & 3 \end{bmatrix} \xrightarrow{\text{swap } r_1 \& r_3} \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ 2 & 3 & 5 \\ \mathbf{0} & \mathbf{1} & \mathbf{2} \end{bmatrix} \xrightarrow{r_2 \leftarrow (r_2 - 2r_1)} \begin{bmatrix} 1 & 2 & 3 \\ \mathbf{0} & \mathbf{-1} & \mathbf{-1} \\ 0 & 1 & 2 \end{bmatrix}$$

$$\underset{r_2}{\xrightarrow{\text{sign change}}} \begin{bmatrix} 1 & 2 & 3 \\ \mathbf{0} & \mathbf{1} & \mathbf{1} \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow[r_3 \leftarrow (r_3 - r_2)]{r_1 \leftarrow (r_1 - 2r_2)} \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{1} \\ 0 & 1 & 1 \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix} \xrightarrow{\substack{r_1 \leftarrow (r_1 - r_3) \\ r_2 \leftarrow (r_2 - r_3)}} \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ 0 & 0 & 1 \end{bmatrix}$$

3B. Are there any conditions b_1, b_2 and b_3 must satisfy in order for $x_1 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ to have a solution? *Explain!* No conditions.

The RREF of C doesn't have any rows-of-all-zeros, so no matter what $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ is, the system $C\vec{x} = \vec{b}$ is consistent.

3C. Explicitly find $C\mathbf{x}$ where $\mathbf{x} = \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}$.

$$C\vec{x} = C\begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix} = 4\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 4 \end{bmatrix} + \begin{bmatrix} 6 \\ 15 \\ 9 \end{bmatrix} = \begin{bmatrix} 6 \\ 23 \\ 13 \end{bmatrix}$$

note well: this question does <u>NOT</u> say:

" Solve $C\vec{x} = \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}$ " !!

(It says, what's $\vec{b}$ if $C\begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix} = \vec{b}$)

4A. What does it mean to say that a set of vectors $S = \{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_p\}$ is *linearly independent*? (Give the definition).

S is L.I. $\Leftrightarrow$

the only solution to $x_1\vec{c}_1 + x_2\vec{c}_2 + \cdots + x_p\vec{c}_p = \vec{0}$

is the trivial soln, $x_1 = x_2 = \cdots = x_p = 0$

4B. In terms of the definition, explain whether the columns of the matrix A from problem 1 form a linearly independent set.

They do NOT. The presence of free variables in $\vec{v}_h$ shows that $A\vec{x} = \vec{0}$ has infinitely many soln's, that is,

$x_1 = x_2 = \cdots = x_5 = 0$ is not the only solution to

$x_1\vec{c}_1 + \cdots + x_5\vec{c}_5 = \vec{0}$ (where $\vec{c}_i$ is the i^{th} column of A.)

4C. In terms of the definition, explain whether the columns of the matrix C from problem 3 form a linearly independent set.

They do. The RREF of $\left[C \,\middle|\, \begin{matrix}0\\0\\0\end{matrix}\right]$ is $\left[\begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix} \,\middle|\, \begin{matrix}0\\0\\0\end{matrix}\right]$,

which means the only soln of $x_1\vec{c}_1 + x_2\vec{c}_2 + x_3\vec{c}_3 = \vec{0}$

is $x_1 = x_2 = x_3 = 0$ (where $\vec{c}_i$'s are the col's of C)

5A. Do the columns of A from problem 1 span $\mathbf{R}^5$? Explain!

No they don't. There are vectors $\vec{b}$ in $\mathbb{R}^5$ which can't be written as L.C.'s of the columns of A because they don't satisfy the conditions found in (1B)

5B. Do the columns of C from problem 3 span $\mathbf{R}^3$? Explain!

yes they do; for any $\vec{b} \in \mathbb{R}^3$, $C\vec{x} = \vec{b}$ has a soln, that is, $\vec{b}$ is a L.C. of the columns of C.

(see (3B))