Here is a fact you may find useful:

For problem 1,

$$\left[\begin{array}{ccccc|ccccc} 0 & 1 & q & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 2 & 3 & r & 7 & 5 & 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & s & 4 & 3 & 0 & 0 & 1 & 0 & 0 \\ 3 & 4 & t & 10 & 8 & 0 & 0 & 0 & 1 & 0 \\ 2 & 2 & u & 6 & 3 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \text{ is row equivalent to } \left[\begin{array}{ccccc|ccccc} 1 & 0 & -3 & 2 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 4 & 1 & 0 & 0 & 0 & 7/4 & -3/4 & 1/4 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1/2 & 1/2 & -1/2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -3/4 & -1/4 & 3/4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -3/4 & -1/4 & -1/4 \end{array}\right]$$

1. Let $A = \begin{bmatrix} 0 & 1 & q & 1 & 2 \\ 2 & 3 & r & 7 & 5 \\ 1 & 2 & s & 4 & 3 \\ 3 & 4 & t & 10 & 8 \\ 2 & 2 & u & 6 & 3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}$ be in $\mathbf{R}^5$, and $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_5$ be the column vectors of A.

Here q, r, s, t and u (the components of $\mathbf{c}_3$) are scalars whose values you will find in problem 2; you do not need their values in problem 1.

1A. Write out the system of equations represented by the matrix equation $A\mathbf{x} = \mathbf{b}$.

1B. Let $\mathcal{S} = \text{Span}\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_5\}$. Find any conditions that $b_1, b_2, \ldots, b_5$ must satisfy so that $\mathbf{b}$ is in $\mathcal{S}$ or state explicitly that there are no special conditions. Note the information on page 0 may be useful!

1C. Suppose that a given $\mathbf{b}$ is indeed in the span of these column vectors. Write the solutions of $A\mathbf{x} = \mathbf{b}$ in the form $\mathbf{p} + \mathbf{v}_h$ where $A\mathbf{p} = \mathbf{b}$ and $A\mathbf{v}_h = \mathbf{0}$. (Note that $\mathbf{p}$ will be in terms of $b_1, b_2, \ldots, b_5$).

1D. (Vocabulary) What are $\mathbf{p}$ and $\mathbf{v}_h$ called?

problem 1, continued.

1E. Show whether the entries of $\mathbf{w} = \begin{bmatrix} 8 \\ 24 \\ 20 \\ 20 \\ 16 \end{bmatrix}$ satisfy all the conditions you found in 1B.

1F. Use (1C) to explicitly solve $A\mathbf{x} = \mathbf{w}$ (use the solution where any free variables are set to 0).

2. Let A be the same matrix as in problem 1, and again let $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_5$ be its column vectors.

2A. Can $\mathbf{c}_5$ be expressed as a linear combination of the other four columns? (Hint: is there a solution of $A\mathbf{x} = \mathbf{0}$ in which $\mathbf{c}_5$ appears with a nonzero weight?) Explain.

2B. Write the zero vector as a linear combination of the columns of A where $\mathbf{c}_3$ has a *nonzero* weight. Make sure your work shows how you found the weights.

2C. Use the answer to (2B) to write $\mathbf{c}_3$ as a linear combination of the other columns of A, and explicitly find the entries q, r, s, t and u of $\mathbf{c}_3$.

3. Let C= $\begin{bmatrix} 0 & 1 & 2 \\ 2 & 3 & 5 \\ 1 & 2 & 3 \end{bmatrix}$

3A. Find the RREF of C *by hand* and label all row operations. You can check your answer by calculator.

3B. Are there any conditions b_1, b_2 and b_3 must satisfy in order for $x_1 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ to have a solution? *Explain!*

3C. Explicitly find $C\mathbf{x}$ where $\mathbf{x} = \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}$.

4A. What does it mean to say that a set of vectors $S = \{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_p\}$ is *linearly independent*? (Give the definition).

4B. In terms of the definition, explain whether the columns of the matrix A from problem 1 form a linearly independent set.

4C. In terms of the definition, explain whether the columns of the matrix C from problem 3 form a linearly independent set.

5A. Do the columns of A from problem 1 span $\mathbf{R}^5$? Explain!

5B. Do the columns of C from problem 3 span $\mathbf{R}^3$? Explain!