INTEGRATION TIPS

- Substitution: usually let \( w \) = an inside function, especially if \( w' \) is also present in the integrand

- Parts: \( \int u \, dv = uv - \int v \, du \) or \( \int uv' \, dx = uv - \int u'v \, dx \)

How to choose which part is \( u \)? Let \( u \) be the part that is higher up in the LIATE mnemonic below. (The mnemonics LIATE and LIPET will work equally well if you have learned one of those instead; in the latter \( A \) is replaced by \( P \), which stands for "polynomial".)

Logarithms (such as \( \ln x \))
Inverse trig (such as \( \arctan x, \arcsin x \))
Algebraic (such as \( x, x^2, x^3 + 4 \))
Trig (such as \( \sin x, \cos 2x \))
Exponentials (such as \( e^x, e^{3x} \))

- Rational Functions (one polynomial divided by another): if the degree of the numerator is greater than or equal to the degree of the numerator, do long division then integrate the result.

Partial Fractions: here's an illustrative example of the setup.

\[ \frac{3x^2 + 11}{(x + 1)(x - 3)(x^2 + 5)} = \frac{A}{x + 1} + \frac{B}{x - 3} + \frac{C}{x^2 + 5} \]

Each linear term in the denominator on the left gets a constant above it on the right; the squared linear factor \((x - 3)\) on the left appears twice on the right, once to the second power. Each irreducible quadratic term on the left gets a linear term \((Dx + E\) here) above it on the right.

1. Find the following.

(a) \( \int \frac{e^{x^2}}{x} \, dx = \int \frac{e^w}{w} \, dw = 2e^w \bigg|_{x=1}^{x=4} = 2e^4 - 2e^1 = 2(e^4 - e^1) \)

(b) \( \int x^3 \ln x \, dx = \frac{x^4}{4} \int \ln x \, dx = \frac{x^4}{4} \left[ x \ln x - \frac{x^2}{2} \right] = \frac{x^4}{4} \left( x \ln x - \frac{x^2}{2} \right) + C \)

(c) \( \int \frac{3x^2 + 2x - 13}{(x - 3)(x^2 + 1)} \, dx = \int \left[ \frac{2}{x - 3} + \frac{x - 1}{x^2 + 1} \right] \, dx = 2 \ln |x - 3| + \frac{1}{2} \ln |x^2 + 1| + \text{Constant} + D \)

Note! If \( u = x^2 + 1 \), then \( du = 2x \, dx \), so

\[ \int \frac{x}{x^2 + 1} \, dx = \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \ln |u| = \frac{1}{2} \ln |x^2 + 1| \]

[Or use #35 in table.]

Partial Fractions: 
\[ \frac{3x^2 + 2x - 13}{(x - 3)(x^2 + 1)} = \frac{A}{x - 3} + \frac{Bx + C}{x^2 + 1} \]

\[ 3x^2 + 2x - 13 = A(x^2 + 1) + (Bx + C)(x - 3) \]

Let \( x = 3 \) \( \Rightarrow 20 = A \cdot 10 + (B \cdot 0 + C)(-3) \Rightarrow A = 2 \)

Let \( x = 0 \) \( \Rightarrow -13 = 2 \cdot 1 + (B \cdot 0 + C)(-3) \Rightarrow -13 = -3C \Rightarrow C = 5 \)

Let \( x = 1 \) \( \Rightarrow -8 = 2 \cdot 2 + (B \cdot 1 + C)(-2) \Rightarrow -8 = 4 - 2B - 10 \Rightarrow B = 1 \)
Long division
\[
\begin{align*}
\text{(d)} & \int \frac{4x^3 - 27x^2 + 20x - 17}{x-6} \, dx = \int \left(4x^2 - 3x + 2 - \frac{5}{x-6}\right) \, dx = \\
& = \frac{4x^3}{3} - \frac{2x^2}{2} + 2x - 5 \ln |x-6| + C \\
\end{align*}
\]

Completing the square:
\[
\begin{align*}
\text{(e)} & \int \frac{2x - 12}{x^2 - 12x + 52} \, dx = \int \frac{\frac{\sqrt{4}}{\sqrt{2}}}{x-6} + 16 \, dx = \\
& = \int \frac{du}{u^2 + 16} = \frac{1}{4} \arctan \left(\frac{x-6}{4}\right) + C \\
\text{Sub: } w &= x-6 \\
\text{dw} &= dx \\
\text{#13 m table}
\end{align*}
\]

Substitution table:
\[
\begin{align*}
\text{(f)} & \int 4x^3 \sin(5x^4) \sin(2x^4) \, dx = \int \frac{\sin(5w) \sin(2w)}{2.3} \, dw = \\
& = \left[ -\frac{\sin(3x^4)}{2.3} + \frac{\sin(7x^4)}{2.7} \right] + C \\
\text{Sub: } w &= x^4 \\
\text{\Rightarrow } dw &= 4x^3 \, dx \\
\text{#44 m table}
\end{align*}
\]

\[
\begin{align*}
\text{(g)} & \text{the area between } y = x^2 - 8x + 24 \text{ and } y = 3x \\
& \text{intersect when} \\
& x^2 - 8x + 24 = 3x \\
& \Rightarrow x^2 - 11x + 24 = 0 \\
& \Rightarrow (x-3)(x-8) = 0 \\
& \Rightarrow x = 3, 8
\end{align*}
\]

2. If \( f(x) \) is decreasing and concave up, put the following quantities in ascending order.

\[
L_{100}, R_{100}, T_{100}, M_{100}, \int_0^b f(x) \, dx
\]

What can you say with certainty about where \( S_{200} \) would fit into your list above?

\[
R_{100} < M_{100} < \int_0^b f(x) \, dx < T_{100} < L_{100}
\]

3. Find the best possible left, right, midpoint, trapezoidal, and Simpson's approximations to \( \int_4^{12} f(x) \, dx \) given the data in the table below.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( 4 )</th>
<th>( 6 )</th>
<th>( 8 )</th>
<th>( 10 )</th>
<th>( 12 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>15</td>
<td>11</td>
<td>8</td>
<td>4</td>
<td>3</td>
</tr>
</tbody>
</table>

\[
L_4 = (15+11+8+4) \cdot 2 = 76 \\
R_4 = (11+8+4+3) \cdot 2 = 52 \\
T_4 = \frac{1}{2}(L_4+R_4) = 64 \\
M_2 = (11 + 4) \cdot 4 = 60 \\
S_4 = \frac{T_4 + 2M_2}{3} = \frac{64 + 2 \cdot 60}{3} = 62 \frac{2}{3}
\]

We cannot compute \( M_4 \); it would require the values \( f(5), f(7), f(9), \) and \( f(11) \). So, we compute \( M_2 \) instead (with \( \Delta x = 4 \)).
4. Find bounds for each of the following errors if \( I = \int_2^7 \ln x \, dx \).

(a) \( |I - L_{100}| \leq \frac{k_1(b-a)^2}{2n} = \frac{1}{2} \cdot \frac{5^2}{200} = \frac{1}{16} \)

\( k_1 = \text{max of } |f'(x)| \text{ on } [2,7] = \text{max of } \frac{1}{x} \text{ on } [2,7] = \frac{1}{2} \text{ (at } x=2 \) \)

(b) \( |I - T_{100}| \leq \frac{k_2(b-a)^3}{12n^2} = \frac{1}{4} \cdot \frac{5^3}{12 \cdot 100^2} = \frac{1}{3840} \)

\( k_2 = \text{max of } |f''(x)| \text{ on } [2,7] = \text{max of } \frac{1}{x^2} \text{ on } [2,7] = \frac{1}{4} \text{ (at } x=2 \) \)

(c) \( |I - M_{100}| \leq \frac{k_3(b-a)^3}{24n^2} = \frac{1}{4} \cdot \frac{5^3}{24 \cdot 100^2} = \frac{1}{7680} \)

\( k_3 = \text{max of } |f'''(x)| \text{ on } [2,7] = \text{max of } \frac{6}{x^3} \text{ on } [2,7] = \frac{6}{16} = \frac{3}{8} \text{ (at } x=2 \) \)

(d) \( |I - S_{100}| \leq \frac{k_4(b-a)^5}{180n^4} = \frac{3}{8} \cdot \frac{5^5}{180 \cdot 100^4} = \frac{1}{1536000} \)

\( k_4 = \text{max of } |f^{(4)}(x)| \text{ on } [2,7] = \text{max of } \frac{6!}{x^5} \text{ on } [2,7] = \frac{6!}{16} = \frac{3}{8} \text{ (at } x=2 \) \)

5. Use Euler's method with three steps on the differential equation \( \frac{dy}{dx} = y - x \) to estimate \( y(2.5) \) if \( y(1) = 0 \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
<th>( \frac{dy}{dx} \cdot \Delta x = \Delta y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>-1.12 = -1</td>
</tr>
<tr>
<td>1.5</td>
<td>-1</td>
<td>-2 = -2</td>
</tr>
<tr>
<td>2</td>
<td>-1.5</td>
<td>-1 = -1</td>
</tr>
<tr>
<td>2.5</td>
<td>-3</td>
<td>-0.12 = -0.15</td>
</tr>
</tbody>
</table>

\( \text{So, } y(2.5) \approx -1.15 \)

6. Find the arc length of \( y = \sqrt{1 - x^2} \) on the interval \([0, 1] \).

\( y' = \frac{1}{2} (1-x^2)^{-1/2} \cdot (-2x) = \frac{-x}{\sqrt{1-x^2}} \), so \( (y')^2 = \frac{x^2}{1-x^2} \)

\( \text{arc length} = \int_0^1 \sqrt{1 + \frac{x^2}{1-x^2}} \, dx = \int_0^1 \sqrt{\frac{1-x^2 + x^2}{1-x^2}} \, dx \) (common denominator)

\( = \int_0^1 \frac{1}{\sqrt{1-x^2}} \, dx = \int_0^1 \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin x \big|_0^1 = \arcsin 1 - \arcsin 0 = \frac{\pi}{2} - 0 = \frac{\pi}{2} \)
7. Consider the region defined by \( y = \sqrt{x}, \ x = 0, \ y = 0, \) and \( x = 9. \) Write an integral equal to the volume generated if this region is rotated about

(a) the \( x- \) axis

\[
\text{vol of slice} \approx \pi r^2 \Delta x = \pi y^2 \Delta x = \pi (\Delta x)^2 \Delta x = \pi x \Delta x
\]

\[
\text{total vol} = \pi \int_0^9 x \, dx
\]

(b) the line \( x = -1 \)

\[
\text{vol of slice} \approx \pi r^2 \Delta y - \pi r^2 \Delta y
\]

\[
\approx \pi 10^2 \Delta y - \pi (1 + x)^2 \Delta y = \pi \left[ 100 - (1 + y^2)^2 \right] \Delta y
\]

\[
\text{total vol} = \pi \int_0^3 \left[ 100 - (1 + y^2)^2 \right] \, dy
\]

8. A pyramid has a square base 30 feet to a side and a height of 10 feet. Write integrals equal to

(a) the volume of the pyramid

\[
\text{cross-section similar triangles:} \quad \frac{10}{30} = \frac{10-h}{s} \Rightarrow s = \frac{3}{10} (10-h) \cdot 3
\]

\[
\text{vol of slice} \approx \pi s^2 \Delta h = \pi \left[ 3 (10-h)^2 \right] dh
\]

\[
\text{total vol} = \int_0^{10} \pi \left[ 3 (10-h)^2 \right] dh
\]

(b) the work done in pumping all the fluid to a point 5 feet above the pyramid if the pyramid is filled to a height of 8 feet with water

\[
\int_0^8 3 (10-h)^2 \cdot 62.4 \cdot (15-h) \, dh
\]

9. Solve the differential equation \( y' = 3(x^2 + 1)(y^2 + 1) \) if the solution passes through the origin.

\[
\frac{dy}{dx} = 3(x^2+1)(y^2+1)
\]

\[
\int \frac{dy}{y^2+1} = \int 3(x^2+1) \, dx
\]

\[
\arctan y = x^3 + 3x + C
\]

\[
y = \tan(x^3 + 3x + C)
\]

\[
y = 0 \text{ when } x = 0, \quad \text{so}
\]

\[
0 = \tan(c) \quad \Rightarrow \quad c = 0
\]

Thus,

\[
y = \tan(x^3 + 3x)
\]