1. Let $\mathbf{a}_1 = \begin{bmatrix} 6 \\ -4 \\ 7 \\ -2 \\ 3 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 5 \\ -6 \\ 3 \\ -3 \\ 2 \end{bmatrix}$, $\mathbf{a}_3 = \begin{bmatrix} -7 \\ 18 \\ 6 \\ 9 \\ -1 \end{bmatrix}$ and $\mathbf{a}_4 = \begin{bmatrix} 3 \\ 2 \\ 1 \\ -1 \\ 2 \end{bmatrix}$. Let $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}$, $\mathbf{m} = \begin{bmatrix} 3 \\ 4 \\ 20 \\ 5 \\ 3 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ 0 \\ 5 \\ -2 \end{bmatrix}$.

Let A be the matrix whose column vectors are $\mathbf{a}_1, \ldots, \mathbf{a}_4$.

It's a fact that $\text{RREF}\left(\left[\begin{array}{c|ccccc} A & 1 & 0 & 0 & 0 & 0 \\ & 0 & 1 & 0 & 0 & 0 \\ & 0 & 0 & 1 & 0 & 0 \\ & 0 & 0 & 0 & 1 & 0 \\ & 0 & 0 & 0 & 0 & 1 \end{array}\right]\right)$ is $\left[\begin{array}{cccc|ccccc} 1 & 0 & 3 & 0 & 0 & 0 & 1/5 & 1/5 & 0 \\ 0 & 1 & -5 & 0 & 0 & 0 & -1/20 & -11/20 & -1/4 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1/4 & 1/4 & 3/4 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1/5 & 4/5 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & -3 & -3 \end{array}\right]$

1A) Find all conditions on $b_1, \ldots, b_5$ that make the equation $A\mathbf{x} = \mathbf{b}$ consistent.

1B) Verify that the entries of $\mathbf{m}$ satisfy the condition(s) in (1A).

1C) Find all values of u_1 and u_2 for which $\mathbf{u}$ is in the span of $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$.

1D) Find all solutions of $A\mathbf{x} = \mathbf{m}$ and express your answer in "parametric vector form" $\mathbf{p} + \mathbf{v}_h$. (Do not be alarmed if $\mathbf{p}$ has some simple fractions in it).

1E) Do the columns of A span $\mathbb{R}^5$? Explain your answer.

1F) Do the columns of A form a linearly independent set? Explain in terms of the definition of linear independent set.

2. Let $C = \begin{bmatrix} 4 & 2 & 2 & 1 \\ 3 & -1 & 9 & 5 \\ 1 & 0 & 2 & 1 \end{bmatrix}$. Let $\mathbf{b}$ be the linear combination of the column vectors of C found by simply adding the last two column vectors of C together.

2A) Explicitly, what is $\mathbf{b}$?

2B) Use RREF($C|\mathbf{b}$) to find all solutions of $C\mathbf{x} = \mathbf{b}$; write your answer in the form $\mathbf{p} + \mathbf{v}_h$ where $\mathbf{p}$ is obtained from the RREF matrix in the usual way, and $\mathbf{v}_h$ is the set of all solutions to the homogeneous equation $C\mathbf{x} = \mathbf{0}$. (Show your RREF here, too)

2C) The statement "Let $\mathbf{b}$ be the linear combination..." at the start of this problem says that $\mathbf{x} = \mathbf{q}$ is a solution of the matrix equation $C\mathbf{x} = \mathbf{b}$ for what particular solution $\mathbf{q}$?

2D) What values do the free variables need to be assigned to satisfy $\mathbf{q} = \mathbf{p} + \mathbf{v}_h$?

2E) Set all the free variables in $\mathbf{p} + \mathbf{v}_h$ to 10. What (new) particular solution do you get (call it $\mathbf{w}$).

2F) Find $C\mathbf{w}$ (hopefully the result isn't a surprise).

2G) The solution set of $C\mathbf{x} = \mathbf{b}$ can also be written as $\mathbf{q} + \mathbf{v}_h$. What values must the free variables have to have in order to satisfy $\mathbf{w} = \mathbf{q} + \mathbf{v}_h$?

3. Let C be the same as in problem (2), so $C = \begin{bmatrix} 4 & 2 & 2 & 1 \\ 3 & -1 & 9 & 5 \\ 1 & 0 & 2 & 1 \end{bmatrix}$.

3A. Find a non-trivial solution $\mathbf{x}$ of $C\mathbf{x} = \mathbf{0}$.

3B. Can you find a non-trivial solution $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ of $C\mathbf{x} = \mathbf{0}$ in which x_4 is non-zero? Explain.

3C. Part (3A) tells us the column vectors $\mathbf{c}_1, \ldots, \mathbf{c}_4$ of C do not form a linearly independent set, and so at least one of the columns is a linear combination of the others. In this case, $\mathbf{c}_3$ is a linear combination of the other three columns. Show this by finding α_1, α_2 and α_4 for which $\mathbf{c}_3 = \alpha_1\mathbf{c}_1 + \alpha_2\mathbf{c}_2 + \alpha_4\mathbf{c}_4$. Show any RREF matrices you use in this problem.

3D. Do the column vectors of C span (all of) $\mathbb{R}^3$? Fully explain your answer. Show any RREF matrices that your answer involves.

4. Suppose the solutions of a matrix equation $A\mathbf{x} = \mathbf{b}$ are written in the form $\mathbf{p} + \mathbf{v}_h$, where $\mathbf{p}$ is a particular solution of $A\mathbf{x} = \mathbf{b}$ and $\mathbf{v}_h$ gives all solutions of the corresponding homogeneous equation.

Suppose $\mathbf{b} = \begin{bmatrix} 2 \\ -13 \\ 0 \\ 4 \end{bmatrix}$, $\mathbf{p} = \begin{bmatrix} 3 \\ 0 \\ 6 \\ -5 \\ 0 \end{bmatrix}$ and $\mathbf{v}_h = x_2 \begin{bmatrix} 8 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -7 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$, where x_2 and x_5 are free.

4A. There is enough information here to determine how many *columns* A has. What is that number of columns?

4B. You can also determine the number of rows in A. How many are there?

4C. How many non-pivot columns does A have?

4D. How many pivot columns does A have?

4E. Is $A\mathbf{x} = \mathbf{d}$ consistent for every $\mathbf{d} \in \mathbb{R}^4$? Explain.

4F. In fact, even though A is not known, you can completely recover (ie, determine all the entries of) RREF($[A|\mathbf{b}]$) from $\mathbf{p} + \mathbf{v}_h$. So, find RREF($[A|\mathbf{b}]$). It may help to first write out explicitly the formulas for each of x_1, x_2, etc, from knowing $\mathbf{p} + \mathbf{v}_h$, and reconstruct RREF($[A|\mathbf{b}]$) from there.