Math 205B&C Quiz 02 page 1 01/30/09 Name Suggested Soln Section: 8am 1:10 pm

Let $A = \begin{bmatrix} 2 & 5 & 10 & -6 & 12 \\ 1 & 2 & 5 & -3 & 6 \\ 3 & 10 & 15 & -8 & 14 \\ 1 & 3 & 5 & -2 & 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 11 \\ 7 \\ 7 \\ 2 \end{bmatrix}$; then RREF of $[A|\mathbf{b}]$ is $\left[\begin{array}{ccccc|c} 1 & 0 & 5 & 0 & -6 & 7 \\ 0 & 1 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & -4 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$.

1A. Use the above information to express all solutions of $A\mathbf{x} = \mathbf{b}$ in the form $\mathbf{p} + \mathbf{v}_h$ where $\mathbf{p}$ is a particular solution of $A\mathbf{x} = \mathbf{b}$ and $\mathbf{v}_h$ represents all solutions of the corresponding homogeneous equation.

we have $\begin{cases} x_1 = 7 - 5x_3 + 6x_5 \\ x_2 = -3 + 0x_3 + 0x_5 \\ x_3 = 1 \cdot x_3 \ (x_3 \text{ is free}) \\ x_4 = -2 + 4x_5 \\ x_5 = 1 \cdot x_5 \ (x_5 \text{ is free}) \end{cases}$ so $\vec{x} = \underbrace{\begin{bmatrix} 7 \\ -3 \\ 0 \\ -2 \\ 0 \end{bmatrix}}_{\vec{p}} + \underbrace{x_3 \begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 6 \\ 0 \\ 0 \\ 4 \\ 1 \end{bmatrix}}_{\vec{v}_h}$, where x_3 & x_5 are free.

1B. In terms of the definition of *linearly independent*, do the columns of A form a linearly independent set? Explain your answer.

In 1A, we see there are non-trivial solutions of $A\vec{x} = \vec{0}$, which means there are solutions to $x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 + x_4\vec{a}_4 + x_5\vec{a}_5 = \vec{0}$ yet not each of x_1, x_2, x_3, x_4, x_5 has to be 0. – this is where the DEFINITION of LI is used.

1C. Label the columns of A as $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_5$. Show explicitly how to express $\mathbf{a}_5$ as a linear combination of the first four columns. Give two different ways to do this, one of which involves a non-zero weight for column $\mathbf{a}_3$, while the other does not use $\mathbf{a}_3$ (ie, its weight is 0).
(Write your answers using the symbols $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_5$; don't copy over all those columns of numbers).

to solve these problems, we'll need solutions of $x_1\vec{a}_1 + \cdots + x_5\vec{a}_5 = \vec{0}$ in which x_5 is non-zero (which will allow us to "solve for" $\vec{a}_5$) Since x_5 is free [see $\vec{v}_h$], we'll choose $x_5 = 1$.

The 1st Question asks for a nonzero weight for $\vec{a}_3$; since x_3 is also free we choose $x_3 = 1$

The other variables become: $x_1 = -5 + 6 = 1$; $x_2 = 0$; $x_4 = 4$ so $1\vec{a}_1 + 0\vec{a}_2 + 1\vec{a}_3 + 4\vec{a}_4 + \vec{a}_5 = \vec{0}$ giving $\boxed{\vec{a}_5 = -\vec{a}_1 - \vec{a}_3 - 4\vec{a}_4}$

The 2nd Question asks for a weight of 0 for $\vec{a}_3$; so we choose $x_3 = 0$.

Now x_1 is $0 + 6 = 6$, $x_2 = 0$, $x_3 = 0$, $x_4 = 4$, $x_5 = 1$, so $6\vec{a}_1 + 4\vec{a}_4 + \vec{a}_5 = \vec{0}$; $\boxed{\vec{a}_5 = -6\vec{a}_1 - 4\vec{a}_4}$

1D. Explain why column $\mathbf{a}_2$ can *not* be written as linear combination of the other four columns.

1 Suppose $\vec{a}_2 = \alpha_1\vec{a}_1 + \alpha_3\vec{a}_3 + \alpha_4\vec{a}_4 + \alpha_5\vec{a}_5$ for scalars $\alpha_1 \alpha_3 \alpha_4 \alpha_5$. Then $\vec{0} = \alpha_1\vec{a}_1 - \vec{a}_2 + \alpha_3\vec{a}_3 + \alpha_4\vec{a}_4 + \alpha_5\vec{a}_5$ and we have a soln to $A\vec{x} = \vec{0}$ in which the weight of $\vec{a}_2$ is -1. This is impossible, as $\vec{v}_h$ shows the weight of $\vec{a}_2$ is 0 in any such solution.

2 OR: Since RREF$([\vec{a}_1 \vec{a}_3 \vec{a}_4 \vec{a}_5 | \vec{a}_2])$ has $[0\,0\,0\,0 | 1]$ as the last row, we cant solve $x_1\vec{a}_1 + x_3\vec{a}_3 + x_4\vec{a}_4 + x_5\vec{a}_5 = \vec{a}_2$

note the vector $\vec{p}$ is NOT involved. otherwise you are writing $\vec{a}_5$ in terms of the other 4 cols. AND $\begin{bmatrix} 11 \\ 7 \\ 7 \\ 2 \end{bmatrix}$!

1E. Do the columns of A span $\mathbb{R}^4$? Why or why not.

NO: Its possible that for some $\vec{b}$'s in $\mathbb{R}^4$, RREF$(A|\vec{b})$ has $[0\,0\,0\,0\,0|1]$ as the last row, showing $A\vec{x} = \vec{b}$ has no soln.