

Math 106: Review for Exam I

1. **Find the following.** [Substitution tip: usually let $u =$ a function that's "inside" another function, especially if du (possibly off by a multiplying constant) is also present in the integrand.]

(a) Let $u = \sqrt{x}$, so $du = \frac{dx}{2\sqrt{x}}$ and $2 du = \frac{dx}{\sqrt{x}}$.

$$\begin{aligned} \int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}} dx &= \int_{x=1}^{x=4} e^u \cdot 2 du && \text{If you prefer to switch the limits, use } u = 1 \text{ to } u = 2. \\ &= 2e^u \Big|_{x=1}^{x=4} \\ &= 2e^{\sqrt{x}} \Big|_1^4 \\ &= 2e^2 - 2e \quad (\approx 9.342) \end{aligned}$$

(b) Let $u = \cos(5x)$, so $du = -5 \sin(5x)$ and $-\frac{du}{5} = \sin(5x)$.

This time, we'll change the limits:

$$x = \pi \Rightarrow u = \cos(5 \cdot \pi) = -1 \text{ and } x = 2\pi \Rightarrow u = \cos(5 \cdot 2\pi) = 1$$

$$\begin{aligned} \int_{\pi}^{2\pi} \cos^7(5x) \sin(5x) dx &= \int_{-1}^1 u^7 \cdot \frac{-du}{5} \\ &= -\frac{1}{5} \int_{-1}^1 u^7 du \\ &= -\frac{1}{5} \frac{u^8}{8} \Big|_{-1}^1 \\ &= -\frac{1}{40} \left[\frac{1^8}{8} - \frac{(-1)^8}{8} \right] \\ &= 0 \end{aligned}$$

(c) Use $u = x^3$, so $du = 3x^2 dx$ and $\frac{du}{3} = x^2 dx$.

$$\begin{aligned} \int \frac{7x^2}{1+x^6} dx &= 7 \int \frac{\frac{du}{3}}{1+u^2} \\ &= \frac{7}{3} \arctan u + C \\ &= \frac{7}{3} \arctan(x^3) + C \end{aligned}$$

- (d) Use #42 from the table of integrals with $n = 3$ and $a = 5$.

$$\begin{aligned} \int \cos^3(5x) dx &= \frac{\cos^2(5x) \sin(5x)}{3 \cdot 5} + \frac{2}{3} \int \cos(5x) dx \\ &= \frac{\cos^2(5x) \sin(5x)}{15} + \frac{2}{3} \int \cos(u) \frac{du}{5} && \text{Substitute } u = 5x, \text{ so } du = 5 dx. \\ &= \frac{\cos^2(5x) \sin(5x)}{15} + \frac{2}{3} \cdot \frac{1}{5} \sin(u) + C \\ &= \frac{\cos^2(5x) \sin(5x)}{15} + \frac{2}{15} \sin(5x) + C \end{aligned}$$

- (e) Use $u = 10 - x$, so $du = -dx$ and $dx = -du$.

We'll change the limits:

$$x = 6 \Rightarrow u = 10 - x = 4 \text{ and } x = 10 \Rightarrow u = 10 - 10 = 0$$

$$\begin{aligned} \int_6^{10} x\sqrt{10-x} dx &= \int_4^0 (10-u)\sqrt{u}(-du) && \text{Since } u = 10 - x, \text{ we know } x = 10 - u. \\ &= \int_4^0 (u-10)\sqrt{u} du \\ &= \int_4^0 (u^{3/2} - 10u^{1/2}) du \\ &= \left[\frac{2}{5}u^{5/2} - \frac{20}{3}u^{3/2} \right]_4^0 \\ &= (0-0) - \left(\frac{2}{5}4^{5/2} - \frac{20}{3}4^{3/2} \right) \\ &= \frac{608}{15} = 40.5\bar{3} \end{aligned}$$

2. If $f(x)$ is decreasing and concave up, put the following quantities in ascending order.

$$L_{100}, R_{100}, T_{100}, M_{100}, \int_a^b f(x) dx \qquad R_{100} < M_{100} < \int_a^b f(x) dx < T_{100} < L_{100}$$

What can you say with certainty about where S_{200} would fit into your list above?

It would be somewhere between M_{100} and T_{100} but we don't know how it compares to $\int_a^b f(x) dx$.

3. Suppose $f(t)$ is the rate of change (in animals per month) of a population $P(t)$.

- (a) What does $\int_4^{12} f(t) dt$ represent in this problem?

It represents the total (or net) change in the number of animals during the time period $[4, 12]$.

- (b) Find the best possible left, right, midpoint, trapezoidal, and Simpson's approximations to $\int_4^{12} f(t) dt$ given the data in the table below.

t	4	6	8	10	12
$f(t)$	15	11	8	4	3

$$L_4 = (15 + 11 + 8 + 4)(2) = 76 \qquad R_4 = (11 + 8 + 4 + 3)(2) = 52 \qquad T_4 = \frac{L_4 + R_4}{2} = 64$$

We cannot compute M_4 because it requires the values of f at $x = 5, 7, 9,$ and 11 . Instead, we do M_2 .

$$M_2 = (11 + 4)(4) = 60 \text{ Now, to find } S_4, \text{ we need } T_2 = \frac{L_2 + R_2}{2} = \frac{(15 + 8)(4) + (8 + 3)(4)}{2} = 68.$$

$$S_4 = \frac{2M_2 + T_2}{3} = \frac{2(60) + 68}{3} = \frac{188}{3} = 62.\bar{6}$$

4. Find bounds for each of the following errors if $I = \int_2^7 \ln x dx$.

$$(a) |I - L_{100}| \leq \frac{K_1(b-a)^2}{2n} = \frac{\frac{1}{2}(7-2)^2}{2(100)} = \frac{1}{16}$$

$$K_1 = \max \text{ of } |f'(x)| \text{ on } [2, 7] = \max \text{ of } \frac{1}{x} \text{ on } [2, 7] = \frac{1}{2} \text{ (occurs at } x = 2)$$

$$(b) |I - T_{100}| \leq \frac{K_2(b-a)^3}{12n^2} = \frac{\frac{1}{4}(7-2)^3}{12(100)^2} = \frac{1}{3840}$$

$$K_2 = \max \text{ of } |f''(x)| \text{ on } [2, 7] = \max \text{ of } \frac{1}{x^2} \text{ on } [2, 7] = \frac{1}{4} \text{ (occurs at } x = 2)$$

$$(c) |I - M_{100}| \leq \frac{K_2(b-a)^3}{24n^2} = \frac{\frac{1}{4}(7-2)^3}{24(100)^2} = \frac{1}{7680}$$

K_2 = same as in previous part

5. If $I = \int_2^7 \ln x \, dx$, how many subdivisions are required to obtain a trapezoidal sum approximation with error of at most $1/1,000,000$?

$$\text{From part (b) above, we know that } |I - T_n| \leq \frac{K_2(b-a)^3}{12n^2} = \frac{\frac{1}{4}(7-2)^3}{12n^2} = \frac{125}{48n^2}.$$

$$\text{Thus, we want } \frac{125}{48n^2} \leq \frac{1}{1,000,000}.$$

$$\text{Multiplying each side by } 1,000,000n^2 \text{ gives } \frac{125,000,000}{48} \leq n^2.$$

$$\text{Taking the square root of each side results in } \sqrt{\frac{125,000,000}{48}} \leq n.$$

Since $\sqrt{\frac{125,000,000}{48}} = 1613.743\dots$, we must at least 1614 subdivisions.

6. Use Euler's method with three steps on the differential equation $\frac{dy}{dt} = y - t$ to estimate $y(2.5)$ if $y(1) = 0$.

t	y	$\frac{dy}{dt} \cdot \Delta t = \Delta y$
1	0	$(-1)(0.5) = -0.5$
1.5	-0.5	$(-2)(0.5) = -1$
2	-1.5	$(-3.5)(0.5) = -1.75$
2.5	-3.25	

So, $y(2.5) \approx -3.25$ (or $-13/4$).

7. Solve the differential equation $dy/dx = 2xy + 6x$ if the solution passes through $(0, 5)$.

$$\frac{dy}{dx} = 2xy + 6x$$

$$\frac{dy}{dx} = 2x(y + 3)$$

$$\frac{dy}{y + 3} = 2x \, dx$$

Separate the variables.

$$\int \frac{dy}{y + 3} = \int 2x \, dx$$

$$\ln|y + 3| = x^2 + C$$

$$|y + 3| = e^{x^2 + C}$$

Exponentiate each side to remove the ln.

$$y + 3 = \pm e^C e^{x^2}$$

$|w| = z$ means $w = \pm z$.

$$y = -3 + Ae^{x^2}$$

Replace $\pm e^C$ with A .

Now we use the initial condition $y(0) = 5$ to find the value of A .

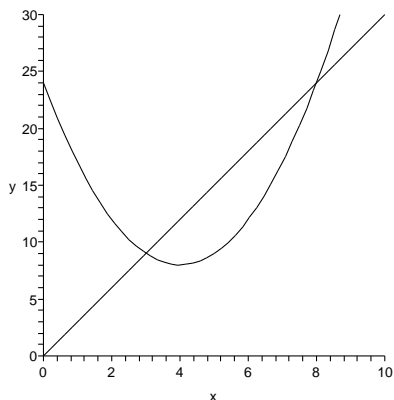
We have $5 = -3 + Ae^0 \Rightarrow A = 8$, so the solution is $y = -3 + 8e^{x^2}$.

8. Write integrals equal to

(a) the arc length of $y = x^2$ on the interval $[1, 5]$

$$\text{arc length of } y = f(x) \text{ on } [a, b] = \int_a^b \sqrt{1 + [f'(x)]^2} dx = \int_1^5 \sqrt{1 + (2x)^2} dx (\approx 24.395)$$

(b) the area bounded by $y = x^2 - 8x + 24$ and $y = 3x$



First, find where the curves intersect.

$$\begin{aligned} x^2 - 8x + 24 &= 3x \\ x^2 - 11x + 24 &= 0 \\ (x - 3)(x - 8) &= 0 \\ \Rightarrow x &= 3, x = 8 \end{aligned}$$

Between $x = 3$ and $x = 8$, $y = 3x$ is above $y = x^2 - 8x + 24$. (Plug in $x = 5$ or graph to check.)

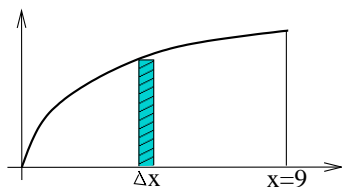
So, the area between them is

$$\int_3^8 [3x - (x^2 - 8x + 24)] dx.$$

[This equals $125/6$.]

9. Consider the region bounded by $y = \sqrt{x}$, $y = 0$, and $x = 9$. Write an integral equal to the volume generated if this region is rotated about

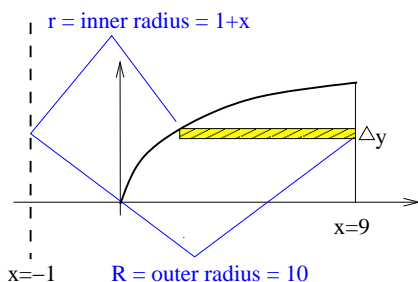
(a) the x -axis



$$\begin{aligned} \text{volume of slice} &\approx \pi r^2 \Delta x \\ &= \pi y^2 \Delta x \\ &= \pi (\sqrt{x})^2 \Delta x \\ &= \pi x \Delta x \end{aligned}$$

$$\text{total volume} = \pi \int_0^9 x dx$$

(b) the line $x = -1$

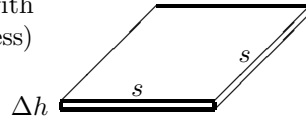


$$\begin{aligned} \text{volume of slice} &\approx \pi R^2 \Delta y - \pi r^2 \Delta y \\ &= \pi (10^2) \Delta y - \pi (1 + x)^2 \Delta y \\ &= \pi [100 - (1 + y^2)^2] \Delta y \end{aligned}$$

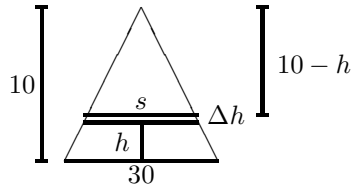
$$\text{total volume} = \pi \int_0^3 [100 - (1 + y^2)^2] dy$$

10. A pyramid has a square base 30 feet to a side and a height of 10 feet. Write an integral equal to the volume of the pyramid.

We slice horizontally, so each slice is a “box” with a square top and bottom and a height (thickness) of Δh , as shown to the right.



The picture shown below is a vertical cross-section through the center of the pyramid.



Similar triangles: $\frac{10}{30} = \frac{10-h}{s} \Rightarrow s = 3(10-h)$.

volume of slice $\approx s^2 \Delta h \approx [3(10-h)]^2 \Delta h$

total volume = $\int_0^{10} [3(10-h)]^2 dh$