

AN ALGORITHMIC APPROACH TO THE DWORK FAMILY

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ABSTRACT. In his work studying the Zeta functions of families of hypersurfaces, Dwork came upon a one-parameter family of hypersurfaces (now known as *the* Dwork family). These examples were not only useful to Dwork in his study of his deformation theory for computing Zeta functions of families, but they have also proven to be extremely useful to physicists working in mirror symmetry. We have developed a computer algorithm, implemented in Pari-GP, which computes the Gauss-Manin connection associated to the Dwork family. A surprising result is that these families are very closely linked to hypergeometric functions. This phenomenon was carefully studied by Dwork and Candelas, de la Ossa, and Rodríguez-Villegas in a few special cases, then later proved by Katz, Rojas-Leon and Wan, and Kloosterman, independently. We verify this latter result by using our algorithm to show that the Picard-Fuchs equation is in fact related to a hypergeometric differential equation and to compute its solutions.

1. INTRODUCTION

In his work studying the Zeta functions of families of hypersurfaces, Dwork came upon a one-parameter family of hypersurfaces in \mathbb{P}^{n-1} (now known as the *Dwork family*), defined by:

$$X_\lambda : x_1^n + \cdots + x_n^n - n\lambda x_1 \cdots x_n = 0.$$

These examples were not only useful to Dwork in his study of his deformation theory for computing Zeta functions of families, but they have also proven to be extremely useful to physicists working in mirror symmetry (c.f. [5]).

In particular, Dwork's work gives a construction of modules isomorphic to the middle (relative) deRham cohomology, equipped with an integrable connection which is equivalent to the Gauss-Manin connection. By the work of Katz and Oda [21], we know that this connection is essentially differentiation of cohomology classes with respect to the parameter. Differentiating each basis element in the module gives us a first-order system of differential equations.

In this paper, we follow the direct approach originally used by Dwork and then Candelas et al. That is, we use Dwork's original construction of a module over \mathbb{C} and algorithmic methods based on the combinatorics and linear algebra inherent to this construction. All the computer algorithms have been implemented in Pari-GP [24] and the GP scripts can be found in the arxiv version of this paper. Our main algorithm computes the matrix for the Gauss-Manin connection associated with the Dwork family, by blocks. This is equivalent to computing the Picard-Fuchs equation for the family. We believe this approach will be useful for the computation of Zeta functions of other families. In fact, Kloosterman in [23] also uses Dwork's cohomology construction to develop a formula that gives the Zeta function associated

to more general monomial deformations of diagonal hypersurfaces. Kloosterman et. al. have also studied the zeta function for certain K3-fibered Calabi-Yau threefolds [13].

Computational methods for studying Zeta functions have been used in many other special cases, like for nondegenerate hypersurfaces with few monomials [29] and for some mirror octics [16]. We believe that the algorithm presented here might be generalizable for a broader class of hypersurfaces, including non-diagonal hypersurfaces in projective space and specific families of hypersurfaces in toric varieties. We especially think that in the context of arithmetic mirror symmetry, i.e. computing Zeta functions of mirror pairs, a computational approach is necessary, as their behavior is not at all well understood. Currently the author is working on modifying this algorithm to study the Zeta function of mirror families of K3-surfaces, beyond the well-understood monomial deformations of the Fermat pencil.

A surprising result is that the Dwork families are very closely linked to hypergeometric functions. This phenomenon was carefully studied by Dwork in the cases where $n = 3, 4$ (see for example [9]) and for $n = 5$ by Candelas, de la Ossa, and Rodríguez-Villegas ([7], [8]). Dwork, Candelas et. al. noticed, in the examples they studied, that the differential equations obtained through this method are hypergeometric differential equations. In recent years, Dwork's ideas have been generalized to compute Zeta functions using p -adic and ℓ -adic cohomology. In studying the Zeta function using ℓ -adic cohomology, Katz proved that there was a link between more general monomial deformations of Fermat hypersurfaces (of which the Dwork family is an example) and hypergeometric sheaves [20]. Rojas-Leon and Wan, independently from Katz, implemented the same approach to compute Zeta-functions [26].

The second algorithm in this paper uses a block of the connection matrix to compute the parameters of the associated hypergeometric differential equation (and in the process proving that this differential equation is hypergeometric). In [23], Kloosterman also shows that the Picard-Fuchs equation associated with the Dwork family is hypergeometric. Here we offer how our algorithm can also lead to this result, in a very straightforward manner.

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2. BACKGROUND

2.1. Hypergeometric Functions.

Definition 2.1. Let, $A, B \in \mathbb{Z}$ and $\alpha_1, \dots, \alpha_A, \beta_1, \dots, \beta_B \in \mathbb{Q}$, with all of the $\beta_i \geq 0$. The **generalized hypergeometric function** is defined as the series (taking $z \in \mathbb{C}$)

$${}_A F_B(\alpha_1, \dots, \alpha_A; \beta_1, \dots, \beta_B | z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_A)_k z^k}{(\beta_1)_k \cdots (\beta_B)_k k!},$$

where we use the Pochhammer notation

$$(x)_k = x(x+1) \cdots (x+k-1) = \frac{\Gamma(x+k)}{\Gamma(x)}.$$

The α_i will be referred to as “numerator parameters” and the β_i as “denominator parameters”.

Sometimes we will use the shortened notation

$${}_A F_B(\alpha; \beta | z) = {}_A F_B(\alpha_1, \dots, \alpha_A; \beta_1, \dots, \beta_B | z).$$

Let θ denote the operator $z \frac{d}{dz}$. The series ${}_A F_B(\alpha; \beta | z)$ satisfies the differential equation

$$\{\theta(\theta + \beta_1 - 1) \cdots (\theta + \beta_B - 1) - z(\theta + \alpha_1) \cdots (\theta + \alpha_A)\} y = 0.$$

Following the notation in [2],

$$D(\alpha_1, \dots, \alpha_A; \beta_1, \dots, \beta_B) = \theta(\theta + \beta_1 - 1) \cdots (\theta + \beta_B - 1) - z(\theta + \alpha_1) \cdots (\theta + \alpha_A).$$

If $A = B + 1$ this is a Fuchsian differential equation with regular singularities at $z = 0, 1, \infty$ (in Section 5 we will review these definitions). We will focus only on hypergeometric functions with this property.

Notice that the parameters α_i, β_i completely characterize the hypergeometric function and its corresponding differential equation.

2.2. Hypergeometric Groups. In Section 5, we will use a certain property of monodromy groups in order to relate the Gauss-Manin connection to hypergeometric functions. First, we need some definitions from [2].

Let H be the fundamental group $\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, z_0)$ where z_0 is some fixed base point, for example $z_0 = \frac{1}{2}$. Then clearly H is generated by g_0, g_1, g_∞ with the relation $g_\infty g_1 g_0 = 1$, as pictured below.

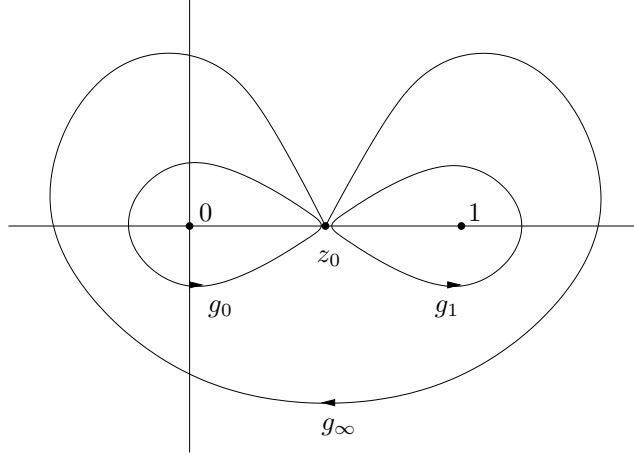
Recall that the differential equation for a hypergeometric function of the form ${}_n F_{n-1}(\alpha; \beta | z)$ is Fuchsian with regular singular points $0, 1, \infty$. Around a regular point, for example $z_0 = \frac{1}{2}$, there are n linearly independent analytic solutions with a non-zero radius of convergence. Let $A, B, C \in \text{GL}(V)$ be determined by analytic continuation of solutions along the generators of $\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, z_0)$, so that

$$\begin{aligned} A &\leftrightarrow g_\infty \\ B &\leftrightarrow g_0 \\ C &\leftrightarrow g_1 \end{aligned}$$

The group $\Gamma \subset \text{GL}(V)$ generated by A, B, C with the relation $ACB = \text{Id}$ is called the *monodromy group*, and the map

$$\begin{aligned} H &\rightarrow \text{GL}(V) \\ g_\infty, g_0, g_1 &\mapsto A, B, C, \end{aligned}$$

is a representation of H .

FIGURE 1. The generators of $\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, z_0)$

Definition 2.2. Let V be a finite dimensional complex vector space. A linear map $g \in \text{GL}(V)$ is called a *reflection* if $g - \text{Id}$ has rank one. The determinant of a reflection is called the *special eigenvalue* of g .

Definition 2.3. Suppose $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{C}^*$ with $a_j \neq b_k$ for all $j = 1, \dots, n$. A *hypergeometric group* with numerator parameters a_1, \dots, a_n and denominator parameters b_1, \dots, b_n is a subgroup of $\text{GL}(n, \mathbb{C})$ generated by elements $h_0, h_1, h_\infty \in \text{GL}(n, \mathbb{C})$ such that $h_\infty h_1 h_0 = \text{Id}$,

$$\det(z - h_\infty) = \prod_{i=1}^n (z - a_i)$$

$$\det(z - h_0^{-1}) = \prod_{j=1}^n (z - b_j),$$

and h_1 is a reflection in the sense of Definition 2.2.

Then we have the following useful result.

Proposition 2.1. [2] Suppose $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{C}^*$ with $a_j \neq b_k$ for all $j, k = 1, \dots, n$ and assume $b_n = 1$. Let $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_{n-1} \in \mathbb{C}$ be such that $a_j = e^{2\pi i \alpha_j}$ for $j = 1, \dots, n$ and $b_k = e^{2\pi i \beta_k}$ for $k = 1, \dots, n-1$. Then the monodromy group of the hypergeometric equation

$$D(\alpha_1, \dots, \alpha_n; \beta_1, \dots, \beta_{n-1})y = 0$$

is a hypergeometric group with parameters $a_1, \dots, a_n, b_1, \dots, b_n$.

Levelt, in his thesis, proved that hypergeometric groups are uniquely determined by the parameters (up to conjugation).

Theorem 2.2 (Levelt). Suppose $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{C}^*$ with $a_j \neq b_k$ for all $j, k = 1, \dots, n$. Let $A_1, \dots, A_n, B_1, \dots, B_n \in \mathbb{C}$ be defined by

$$\prod_{j=1}^n (t - a_j) = t^n + A_1 t^{n-1} + \cdots + A_n, \quad \prod_{j=1}^n (t - b_j) = t^n + B_1 t^{n-1} + \cdots + b_n$$

and let $A, B \in GL(n, \mathbb{C})$ be given by

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 & -A_n \\ 1 & 0 & \cdots & 0 & -A_{n-1} \\ 0 & 1 & \cdots & 0 & -A_{n-2} \\ & & \ddots & & \\ 0 & 0 & \cdots & 1 & -A_1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & \cdots & 0 & -B_n \\ 1 & 0 & \cdots & 0 & -B_{n-1} \\ 0 & 1 & \cdots & 0 & -B_{n-2} \\ & & \ddots & & \\ 0 & 0 & \cdots & 1 & -B_1 \end{pmatrix},$$

Then the matrices $h_\infty = A, h_0 = B^{-1}, h_1 = A^{-1}B$ generate a hypergeometric group with parameters $a_1, \dots, a_n, b_1, \dots, b_n$. Moreover, any hypergeometric group with the same parameters is conjugated inside $GL(n, \mathbb{C})$ to this one.

Remark 2.1. The most important consequence of the previous two results is that if we have a hypergeometric group Γ in the sense of the Definition 2.3, we can find a hypergeometric differential equation whose monodromy group is Γ , with parameters determined by the eigenvalues of h_0 and h_∞ .

3. DWORK'S CONSTRUCTION

Suppose X is a hypersurface in \mathbb{P}^{n-1} , and so it is $n - 2$ complex dimensional. The *middle* deRham cohomology will be the $(n - 2)$ -th cohomology. It is a classical result by Lefschetz that the i -th deRham cohomology of a non-singular projective hypersurface of dimension $n - 2$ is identical to the i -th deRham cohomology of \mathbb{P}^{n-1} , for $i \neq n - 2$. In other words, the middle cohomology is the only “interesting” one.

Recall the Dwork family of hypersurfaces defined by

$$X_\lambda : x_1^n + \cdots + x_n^n - n\lambda x_1 \cdots x_n = 0.$$

It is not difficult to see that X_λ is not smooth if and only if λ is an n -th root of unity. Let $T = \mathbb{C} - \mu_n$. It follows that X_λ is non singular for $\lambda \in T$. Dwork constructed modules over \mathbb{C} isomorphic to the relative deRham cohomology $H_{dR}^{n-2}(X_\lambda/T)$, which are quite combinatorial in nature.

To give Dwork's construction, we will use the notation established in [18]. Let $F_\lambda(x_1, \dots, x_n) := x_1^n + \cdots + x_n^n - n\lambda x_1 \cdots x_n$. Let \mathfrak{L} be the free module (over $\mathbb{C}(\lambda)$) generated by the monomials

$$x_1^{w_1} \cdots x_n^{w_n} = x^w,$$

with all the $w_i \geq 0$ and $\sum_{i=1}^n w_i \equiv 0 \pmod{n}$.

Let \mathfrak{L}^S be the submodule generated by monomials x^w with all $w_i \geq 1$. Let D_i be the mapping defined by

$$D_i : \mathfrak{L} \rightarrow \mathfrak{L}, \quad D_i(x^w) = w_i x^w + x_i \frac{\partial F_\lambda}{\partial x_i} x^w.$$

So $D_i(x^w) = w_i x^w + n x_i^n x^w - n\lambda x_1 \cdots x_n x^w$.

We now define the $\mathbb{C}(\lambda)$ -module

$$\mathcal{W} = \mathfrak{L}^S / \mathfrak{L}^S \cap \left(\sum_{i=1}^n D_i \mathfrak{L} \right).$$

In fact, \mathcal{W} is a vector bundle over T .

Notation. We will frequently represent a monomial x^w by its exponent w and the notation cw signifies cx^w , where $c \in \mathbb{C}(\lambda)$.

In this notation, observe that the space \mathcal{W} has n relations of the form:

$$w_i(w_1, \dots, w_n) + n(w_1, \dots, w_i + n, \dots, w_n) - n\lambda(w_1 + 1, \dots, w_n + 1) = 0$$

coming from the quotient by the D_i defined above.

This can be rearranged as

$$(w_1, \dots, w_n) = \frac{n - w_i}{n}(w_1, \dots, w_i - n, \dots, w_n) + \lambda(w_1 + 1, \dots, w_i - n + 1, \dots, w_n + 1).$$

Notice that this implies that any monomial with a power larger than n can be written in terms of two other monomials in \mathcal{W} . In fact, this is going to give us a way to reduce any monomial with powers larger than n to a linear combination of monomials with powers between 1 and $n - 1$. That is the content of the following Algorithm.

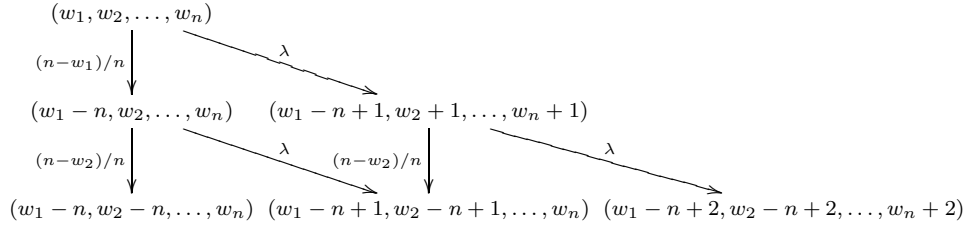
Algorithm 1 (Reduction Algorithm). *We define an algorithm on n -tuples w using the relations given by the quotient described above. The input for this algorithm is $cw = c(w_1, w_2, \dots, w_n)$, where c is an element of $\mathbb{C}(\lambda)$, and w is an n -tuple representing a monomial in \mathcal{W} where $w_i \geq n$ for some i . The output is a list that represents how to write cw as a linear combination of monomials for which all exponents are less than n , thus “reducing” w .*

- (1) Initialize two empty lists, L and M .
- (2) Let i be the first entry such that $w_i \geq n$. Define $u = c\lambda(w_1 + 1, w_2 + 1, \dots, w_i - n + 1, \dots, w_n + 1)$ and $v = c\frac{n-w_i}{n}(w_1, \dots, w_i - n, \dots, w_n)$. If $\frac{n-w_i}{n} \neq 0$:
 - (a) If $v_i < n$ for all i , append v to list L , unless $(w_1, \dots, w_i - n, \dots, w_n)$ is already in the list, then add $c\frac{n-w_i}{n}$ to the existing coefficient. Go to step 3.
 - (b) If $v_i \geq n$ for some i , append v to list M , unless $(w_1, \dots, w_i - n, \dots, w_n)$ is already in the list, then add $c\frac{n-w_i}{n}$ to the existing coefficient. Go to step 3.
- (3) If $u_i < n$ for all i , append u to L and proceed to step 4 (here if u is already in L we add coefficients as before). Otherwise, proceed to step 5.
- (4) If the list M is empty, **output** the list L . Otherwise, proceed to step 8.
- (5) If $u = w$, proceed to step 6. Otherwise, proceed to step 7.
- (6) If the list M is empty, **output** the list L but with all coefficients divided by $1 - \lambda^n$. Otherwise, proceed to step 8.
- (7) Append u to M and proceed to step 8.
- (8) Let a be the first element in M . Apply step 2 to a instead of w and remove a from M .

Proof. Notice that we stop the algorithm whenever the list M is empty. The list M contains monomials with entries that are greater than or equal to n . To prove the algorithm terminates we need to prove that using this reduction process we can always empty the list.

Given any starting n -tuple w , the worst that can happen is that every entry is greater than or equal to n . In this case, the first step would be to set $u = c\lambda(w_1 - n + 1, w_2 + 1, \dots, w_n + 1)$ and $v = c\lambda^{\frac{n-w_1}{n}}(w_1 - n, \dots, w_n)$. At most two monomials get added to M (v is added only if its coefficient is nonzero). We remove the first monomial (suppose this is u) and apply the reduction again. So $u_2 = c\lambda^2(w_1 - n + 2, w_2 - n + 2, \dots, w_n + 2)$ and $v_2 = c\lambda^{\frac{n-w_2}{n}}(w_1 - n + 1, w_2 - n + 1, \dots, w_n + 1)$. Once more, at most two monomials get added to M . But now notice that the next item on the list would be v . Applying the reduction to this monomial would give $u_3 = c\lambda^{\frac{n-w_1}{n}}(w_1 - n + 1, w_2 - n + 1, \dots, w_n + 1)$ and $v_3 = c\lambda^{\frac{n-w_1}{n}}\lambda^{\frac{n-w_2}{n}}(w_1 - n, w_2 - n, \dots, w_n)$. Notice that $u_3 = kv_2$, where k is some constant, and thus for each pair of monomials we remove we are only really adding, at most, the number of monomials we had before plus one.

One can visualize the process in a diagram as follows:



In the tree above, there are two types of arrows, which we will denote “down” and “right”. Notice the down arrows indicate the operation of subtracting n from the first position that is greater than n , and this process clearly terminates. That is, all of the columns in the tree terminate eventually. The only concern, then, is whether the top “right” portion of the tree terminates.

Now notice that the right arrows involve subtracting n from the first entry larger than n and then adding 1 to all entries. If this process does not terminate in a monomial whose entries are all less than n then at the n -th reduction step we will find the monomial $(w_1 - n + n, w_2 - n + n, \dots, w_n - n + n) = w$. Either way, the top right portion of the tree has to end, and the algorithm finishes once we have emptied M . But “emptying M ” is equivalent to traveling through the whole tree, which we now know is finite. Thus, the algorithm terminates. \square

Corollary 3.1. \mathcal{W} is generated over $\mathbb{C}(\lambda)$ by the set of monomials

$$\mathcal{B} = \left\{ x_1^{w_1} \cdots x_n^{w_n} = x^w \mid 1 \leq w_i \leq n-1, \sum w_i \equiv 0 \pmod{n} \right\}.$$

In particular, \mathcal{W} has dimension

$$(n-1)^{(n-1)} - (n-1)^{(n-2)} + (n-1)^{(n-3)} - \cdots \pm (n-1).$$

Proof. The first statement is a direct consequence of Algorithm 1.

For the second statement, it is a short calculation to verify that \mathcal{B} has

$$(n-1)^{(n-1)} - (n-1)^{(n-2)} + (n-1)^{(n-3)} - \cdots \pm (n-1)$$

elements.

The last thing one needs to show is that these monomials are linearly independent, and thus form a basis for \mathcal{W} . For this, we appeal to Lemma 3.19 in [11]. The dimension of the space in Dwork’s construction is exactly the same as the one we get from the generating set \mathcal{B} , and thus our set must be linearly independent. \square

The vector bundle \mathcal{W} is equipped with an integrable connection ∇ defined by

$$\nabla(f(\lambda)x^w) = \frac{\partial}{\partial \lambda} f(\lambda)x^w + f(\lambda) \frac{\partial}{\partial \lambda} F_\lambda x^w.$$

In particular, for monic monomials,

$$\nabla(x^w) = \frac{\partial}{\partial \lambda} F_\lambda x^w = -nx \cdot x^w.$$

Katz, in [18], proved the following useful lemmas.

Theorem 3.2 (The Comparison Theorem). *Let $w_0 = \frac{1}{n} \sum_{i=1}^n w_i$. There is a T -linear map $\mathcal{R} : \mathfrak{L}^S \rightarrow H_{dR}^{n-1}(\mathbb{P}^n - X_\lambda/T)$ given by*

$$\mathcal{R} : x^w \mapsto (-1)^{w_0} (w_0 - 1)! \frac{x^w}{F_\lambda^{w_0}} \frac{d(x_1/x_n)}{x_1/x_n} \wedge \cdots \wedge \frac{d(x_{n-1}/x_n)}{x_{n-1}/x_n}.$$

By this theorem and the residue map (cf. [14]) we have an isomorphism from $H_{dR}^{n-1}(\mathbb{P}^n - X_\lambda/T)$ to $H_{dR}^{n-2}(X_\lambda/T)$.

And we have the following:

Theorem 3.3. *The map Θ induces, by passage to quotients, an isomorphism*

$$\Theta : \mathcal{W} \xrightarrow{\sim} H_{dR}^{n-2}(X_\lambda/T),$$

which is compatible with the connection.

Hence the space \mathcal{W} obtained through Dwork's construction is isomorphic to the middle (relative) deRham cohomology.

It is also important to note that Θ transforms ∇ into the Gauss-Manin connection.

4. COMPUTING THE CONNECTION MATRIX

Let μ_n^n denote the group of n -tuples of n -th roots of unity, and Δ denote the diagonal elements.

The character group of μ_n^n/Δ is in one-to-one correspondence with the set

$$W = \{(w_1, \dots, w_n) | 0 \leq w_i < n, \sum w_i \equiv 0 \pmod n\},$$

where

$$\chi_w(\xi) := \chi(\xi^w), \quad \xi^w = \xi_1^{w_1} \cdots \xi_n^{w_n}$$

and χ is a fixed primitive character of μ_n .

Let

$$G = \{\xi \in \mu_n^n | \xi_1 \cdots \xi_n = 1\} / \Delta.$$

The characters χ_w of μ_n^n/Δ which act trivially on G are precisely powers of $\chi_{\bar{1}}$, where $\bar{1} = (1, 1, \dots, 1)$. Thus, $\text{Char}(G)$, the character group of G , corresponds to equivalence classes of w in W , where $w' \sim w$ if $w - w'$ is a multiple (mod n) of $\bar{1}$.

The family X_λ allows a faithful action of the group G by $\xi = (\xi_1, \dots, \xi_n)$ taking the point (x_1, \dots, x_n) to $(\xi_1 x_1, \dots, \xi_n x_n)$. Using this action, we get that \mathcal{W} splits into eigenspaces as follows.

Proposition 4.1. *The action of G on \mathcal{W} gives*

$$\mathcal{W} = \bigoplus_{\chi \in \text{Char}(G)} \mathcal{W}(\chi),$$

where $\mathcal{W}(\chi)$ is an eigenspace with basis

$$\{w, w + \bar{1}(\text{mod } n), \dots, w + \overline{n-1}(\text{mod } n)\},$$

but we exclude adding any vector \overline{m} such that $m + w_i \equiv 0 \pmod n$ for some i .

To understand ∇ 's effect on \mathcal{W} , it is sufficient to know what it does to elements in the basis \mathcal{B} . From the definition of ∇ we see that

$$\nabla(x^w) = \frac{\partial}{\partial \lambda} F_\lambda x^w = -n x^{w+\bar{1}}$$

where $w + \bar{1} = (w_1 + 1, \dots, w_n + 1)$.

Thus, the connection commutes with the action of G , so the proposition implies that ∇ preserves eigenspaces. We want to compute the connection matrix ∇ . Because of the way in which ∇ preserves eigenspaces, the connection matrix will have blocks on its diagonal for each eigenspace.

The main idea of the following algorithm is to use the reduction algorithm described earlier on $\nabla(x^w)$ where w runs through the basis of an eigenspace.

Algorithm 2 (Computing a block of the connection matrix). *This algorithm takes any vector of integers as an input and outputs a matrix that is the block of the connection matrix that corresponds to that vector's eigenspace generators.*

- (1) Create a basis of the eigenspace related to w by computing $\mathcal{B} = \{w, w + \bar{1}(\text{mod } n), \dots, w + \overline{n-1}(\text{mod } n)\} = \{v_1, \dots, v_k\}$, where we omit monomials which have entries equal to 0 mod n (so k may or may not equal n).
- (2) Create M , a $k \times k$ matrix of zeros.
- (3) Let $i = 1$.
- (4) If $i = k+1$, **output** M . Otherwise, take the monomial v_i in \mathcal{B} and compute its derivative, that is $\nabla(v_i) = -n(v_i + \bar{1})$.
- (5) If all the entries in $\nabla(v_i)$ are less than n , then add $-n$ to the $(i, i+1)$ position in M , add 1 to i , and go back to Step 4. If not, proceed to the next step.
- (6) Reduce $\nabla(v_i)$ according to Algorithm 1. From the algorithm, we have $\nabla(v_i)$ as a linear combination of elements of \mathcal{B} , so that $\nabla(v_i) = \alpha_1 v_1 + \dots + \alpha_k v_k$. Replace the i -th row of M by $(\alpha_1, \dots, \alpha_k)$. Add 1 to i and go to Step 4.

5. THE DIFFERENTIAL EQUATION ASSOCIATED TO THE CONNECTION

In this section, we will show that the differential equation associated to the connection ∇ is related to a hypergeometric differential equation. We have developed an algorithm which outputs the parameters α, β given each block representative.

We will first establish some notation and definitions. For more details and proofs, see for example [3], [6], or [15].

Consider the n th order equation

$$(1) \quad \sum_{m=0}^n a_{n-m}(z) y^{(m)} = 0, \quad (a_0(z) \equiv 1)$$

where the $a_k(z)$ are single-valued and analytic in a punctured neighborhood of a point z_0 . Recall that if any of the a_k have a singularity at z_0 , then z_0 is called a *singular point* for (1), otherwise it is called an analytic point. We say z_0 is a *regular singular point* if

$$(2) \quad a_k(z) = (z - z_0)^{-k} b_k(z), \quad (k = 1, \dots, n),$$

where b_k is analytic at z_0 .

A system of n first order equations over $\mathbb{C}(z)$ has the form

$$(3) \quad y' = Ay$$

in the unknown column vector $y = (y_1, \dots, y_n)^T$ and where A is an $n \times n$ -matrix with entries in $\mathbb{C}(z)$. The entries are assumed to be single-valued and analytic at a neighborhood of a point z_0 , and will at most have a pole at that point.

If A has a singularity at z_0 , then z_0 is a *singular point* for the system (3). z_0 is a *regular singular point* if

$$A(z) = (z - z_0)^{-1} \tilde{A}(z)$$

where \tilde{A} is analytic for a neighborhood of z_0 (including z_0), and $\tilde{A}(z_0) \neq 0$.

A differential system or a differential equation for which all singularities are regular is called *Fuchsian*.

To study the system at ∞ , we change variables from z to $1/\zeta$. The associated system is

$$\frac{d\tilde{y}}{d\zeta} = -\frac{\tilde{A}(\zeta)}{\zeta^2} \tilde{y}.$$

It is not difficult to see that a differential equation like (1) can be rewritten as a system by setting $y_1 = y, y_2 = y', \dots, y_n = y^{(n-1)}$. Notice that this means $y'_1 = y_2, y'_2 = y_3, \dots, y'_{n-1} = y_n$, and y'_n is given by the differential equation. So the differential system is determined by a companion matrix, as follows:

$$(4) \quad \frac{d}{dz} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}.$$

Now, we will explain how a system of first order differential equations arises from the connection. Recall that, on a vector bundle, being equipped with a connection ∇ is equivalent to being equipped with an action of $\frac{d}{d\lambda}$ (see for example [22] for a detailed explanation). In short, we have a first-order system defined by

$$\frac{d}{d\lambda} y = Ay,$$

where A is actually the transpose of the matrix we found in the previous section. In fact, each block defines its own differential system. We will use the following fact:

Theorem 5.1 (Cyclic Vector Lemma). *Any system of linear first order differential equations is equivalent to a system which comes from a differential equation.*

A proof of the lemma can be found in [12].

Basically, this theorem says that in the space of solutions of our differential system there is a cyclic vector, that is, a vector such that

$$v, \frac{dv}{d\lambda}, \frac{d^2v}{d\lambda^2}, \dots, \frac{d^{n-1}v}{d\lambda^{n-1}}$$

is a basis. More specifically, this means that if $y = (y_1, y_2, \dots, y_n)^T$ is a solution for the system, we can find an equivalent system with solutions of the form $\hat{y} = \left(\hat{y}_1, \frac{d\hat{y}_1}{d\lambda}, \dots, \frac{d^{n-1}\hat{y}_1}{d\lambda^{n-1}} \right)^T$. In fact, using the system, we can represent the derivatives $\hat{y}_1^{(k)}$ as a linear combination of y_1, \dots, y_n . This determines a change of basis matrix S such that $Sy = \hat{y}$.

The vector $Sy = \hat{y}$ satisfies a differential system of the form

$$\frac{d}{d\lambda}\hat{y} = \left(SAS^{-1} + \frac{dS}{d\lambda}S^{-1} \right) \hat{y},$$

and this last system is the companion matrix to a higher order differential equation. In our situation, since the basis vectors are basically already derivatives of each other, any vector in the basis, for example y_1 , is a cyclic vector, and so S is easy to determine. Let $A_S = \left(SAS^{-1} + \frac{dS}{d\lambda}S^{-1} \right)$.

Notice that, from the reduction algorithm, the entries in the connection matrix are rational functions in λ . Furthermore, $1 - \lambda^n$ is the only possible non-trivial denominator. After doing the change of basis described above, we may get some powers of λ in the denominator, but as the new system is a companion matrix, this will only happen in the last row of the matrix.

At this point, we have a differential equation associated to the connection. But solving high order differential equations is not a simple task. Instead, we will show that these matrices are related to a hypergeometric group, which in turn gives us the defining parameters of the hypergeometric differential equation we want to find.

We have a way of changing from a differential equation to a system and viceversa. It is important to note that a regular singular point of (1) z_0 *may not* be a regular singular point of the system associated with it. This happens only when the a_k have at most simple poles at z_0 . However, there is an equivalent first-order system with the property that if z_0 is a regular singular point of (1) then z_0 is a regular singular point of the system.

Suppose (1) has a regular singularity at z_0 , and let ϕ be a solution of (1). Define $\hat{\phi}$ to be the vector with components ϕ_1, \dots, ϕ_n by setting

$$\phi_k = (z - z_0)^{k-1} \phi^{(k-1)}, \quad (k = 1, \dots, n).$$

Then, clearly,

$$\begin{aligned} (z - z_0)\phi'_k &= (z - z_0)((z - z_0)^{k-1}\phi^{(k-1)})' \\ &= (z - z_0)((k-1)(z - z_0)^{k-2}\phi^{(k-1)} + (z - z_0)^{k-1}\phi^{(k)}) \\ &= (k-1)(z - z_0)^{k-1}\phi^{(k-1)} + (z - z_0)^k\phi^{(k)} \\ &= (k-1)\phi_k + \phi_{k+1} \quad (k = 1 \dots, n-1) \end{aligned}$$

And, finally,

$$(z - z_0)\phi'_n = (n - 1)\phi_n - \sum_{m=1}^n b_{n-m+1}(z)\phi_m,$$

where the b_i are defined as in (2).

Therefore $\hat{\phi}$ is a solution of the linear system

$$(5) \quad y' = \hat{A}(z)y$$

where \hat{A} has the structure

$$\hat{A}(z) = (z - z_0)^{-1} \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 1 & \cdots & 0 \\ 0 & 0 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -b_n & -b_{n-1} & -b_{n-2} & -b_{n-3} & \cdots & (n-1) - b_1 \end{pmatrix}$$

This system clearly has a regular singularity at z_0 .

In our situation, we want a hypergeometric differential system, so in particular we want a simple pole at $\lambda = 0$. We can accomplish this by replacing the matrix A_S by the matrix \hat{A} , which has a simple pole at 0. In particular, now we are certain that the only denominators in the last row are of the form $1 - \lambda^n$.

Recall that a hypergeometric differential system must only have regular singularities at 0, 1, and ∞ . The previous step took care of 0 and ∞ , but we have singularities at all n -th roots of unity, not just 1. We will deal with this by changing variables to $z = \lambda^n$. We will first discuss how this changes our differential system. By the chain rule, we have:

$$\frac{d}{d\lambda}y = \frac{d}{dz}y \frac{d}{d\lambda}z = \frac{d}{dz}y * n\lambda^{(n-1)}$$

The system representing the derivative with respect to z is $y' = By$ where $B = \frac{1}{n\lambda^{(n-1)}}\hat{A}$. Since \hat{A} has a simple pole at $\lambda = 0$, replacing every instance of λ^n by z gives that B has a simple pole at $z = 0$ as well, but now we also have a simple pole at $z = 1$. So, in fact, the system satisfied by the Gauss-Manin connection is the pullback by the n -th power map of the new system involving B . We will proceed to solve $y' = By$, as this will be the hypergeometric system we want.

There is an algorithm by Brieskorn which relates Gauss-Manin connections to monodromy group generators [4]. Let A be the matrix representation of the connection. The algorithm uses the fact that if A has a simple pole around a given point, i.e. can be written as

$$A = A_{-1}(z - z_0)^{-1} + A_0 + A_1(z - z_0) + \cdots$$

then the eigenvalues of $R = e^{2\pi i A_{-1}}$ are the eigenvalues for the monodromy around z_0 .

The process of changing the system matrix to B ensures that we have a simple pole around zero, one, and infinity. Since this last system has regular singular points at 0, 1, ∞ and no other singularities, it is Fuchsian, as we expected.

Let B_{-1} denote the residue around zero, and \tilde{B}_{-1} denote the residue at ∞ . Just as above, the eigenvalues of $h_0 = e^{2\pi i B_{-1}}$ and the eigenvalues of $h_\infty = e^{2\pi i \tilde{B}_{-1}}$ give the eigenvalues of the monodromies around 0 and ∞ respectively. Let D be the residue of B around $z = 1$. One can see by looking at the eigenvalues of $h_1 = e^{2\pi i D}$ that it is clearly a reflection in the sense described by Beukers and Heckman. This is easily checked by noticing that there is only one row with denominators of the form $1 - z$. Thus, the residue at one will necessarily have rank one, which implies that $h_1 - \text{Id}$ has rank one.

Recall that by Remark 2.1, a hypergeometric group is uniquely determined, up to common conjugation, by the eigenvalues of the monodromies. Thus, we have shown that the eigenvalues of the matrices h_∞, h_1, h_0 define a hypergeometric group, associated to a unique hypergeometric differential equation where the eigenvalues of B_{-1} will be the β 's and the eigenvalues of \tilde{B}_{-1} will be the α 's. Therefore, the monodromy group we have found corresponds to the differential equation

$$D(\alpha; \beta|z)y = 0,$$

where the α 's are the eigenvalues of \tilde{B}_{-1} and the β 's are the eigenvalues of B_{-1} .

Now, recall that the original differential equation satisfied by the connection is a pullback by the n -th power map of the one we just found. So in fact, we get that the hypergeometric function associated to the differential equation satisfied by the Gauss-Manin connection is

$${}_nF_{n-1}(\alpha; \beta|\lambda^n)y = 0.$$

This completes our algorithmic proof that every block of the Gauss-Manin connection for the Dwork family is related to a hypergeometric function.

Here is the algorithm we have just described.

Algorithm 3 (Computing the parameters of the hypergeometric differential equation). *This algorithm takes as input a monomial w and computes the parameters of the hypergeometric differential equation associated to the eigenspace generated by w . The output is two lists, α and β , containing the parameters.*

- (1) Let A be the $k \times k$ connection matrix block associated to w , computed with Algorithm 2.
- (2) Compute the change of basis matrix, S , given by assuming w as a cyclic vector.
- (3) Compute $A_S = \left(SAS^{-1} + \frac{dS}{d\lambda} S^{-1} \right)$.
- (4) Compute \hat{A} by multiplying $a_{k,j}$ by λ^{k-j+1} and adding $i - 1$ in the (i, i) position of A_S for $i = 1, \dots, k$. This means that in the (k, k) position we have $k - 1 + \lambda a_{k,k}$.
- (5) Multiply \hat{A} by $1/n$, call this B .
- (6) Let $h_0 = \text{Res}_{z=0} B$ and compute the eigenvalues. Let β be the list of eigenvalues.
- (7) Let $h_\infty = \text{Res}_{z=\infty} B$ and compute the eigenvalues. Let α be the list of these eigenvalues.
- (8) **Output:** α, β .

Remark 5.1. While computing some examples, we noticed that given a vector (w_1, w_2, \dots, w_n) , if we cancel out the numbers which it has in common with the

list $(0, 1, 2, \dots, n-1)$, then $\alpha_i = \frac{w_j}{n}$ for each w_j that survives the cancelation, and $\beta_i = \frac{k}{n}$ for each k that survives in the second vector. This is exactly Katz's main result in the case of hypergeometric sheaves [20].

6. AN ILLUSTRATIVE EXAMPLE

Suppose $n = 6$. By Algorithm 1 we have that

$$\mathcal{W} = \left\langle x^w \mid 1 \leq w_i \leq 5, \sum w_i \equiv 0 \pmod{6} \right\rangle,$$

and \mathcal{W} has dimension $5^5 - 5^4 + 5^3 - 5^2 + 5 = 2605$.

Take $w = (1, 1, 1, 2, 2, 5)$. This belongs to the eigenspace generated by

$$\mathcal{B}_{(1,1,1,2,2,5)} = \{(1, 1, 1, 2, 2, 5), (3, 3, 3, 4, 4, 1), (4, 4, 4, 5, 5, 2)\}.$$

Here is an example of the algorithm for computing the block in the matrix representation of ∇ for $n = 6$ and the eigenspace corresponding to the monomial $(1, 1, 1, 2, 2, 5) \sim x_1 x_2 x_3 x_4^2 x_5^2 x_6^5$, with basis denoted earlier by $\mathcal{B}_{(1,1,1,2,2,5)}$. I will denote this block by $\nabla_{\mathcal{B}_{(1,1,1,2,2,5)}}$.

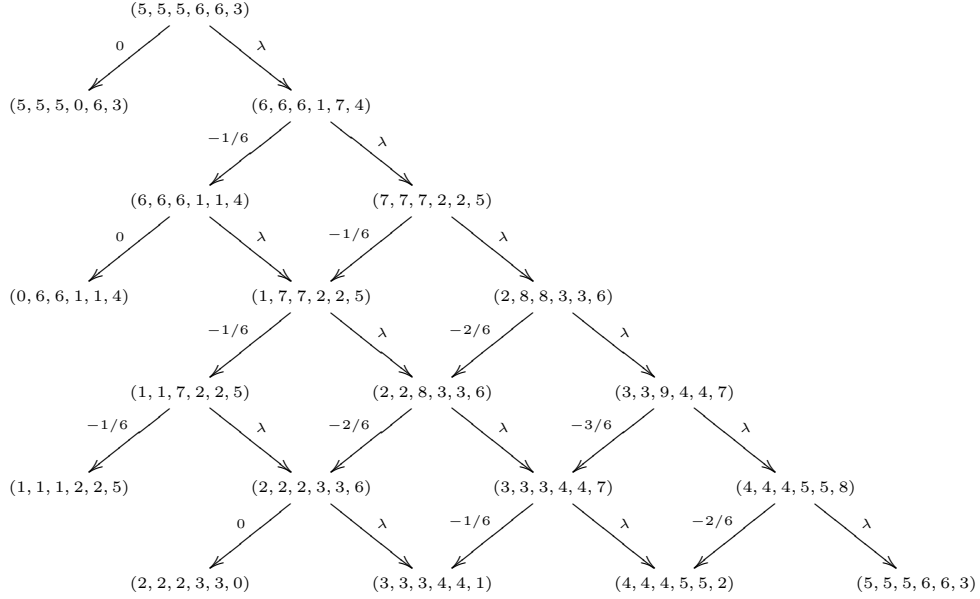
- (1) Apply $\nabla(1, 1, 1, 2, 2, 5) = -6(2, 2, 2, 3, 3, 6)$. Using the relations we can write this last monomial in terms of the monomials in $\mathcal{B}_{(1,1,1,2,2,5)}$. We can represent the process of Algorithm 1 graphically, as shown below:

$$\begin{array}{ccc} & (2, 2, 2, 3, 3, 6) & \\ \swarrow 0 & & \searrow \lambda \\ (2, 2, 2, 3, 3, 0) & & (3, 3, 3, 4, 4, 1) \end{array}$$

This means that $(2, 2, 2, 3, 3, 6) = \lambda(3, 3, 3, 4, 4, 1) + 0 \cdot (2, 2, 2, 3, 3, 0)$, which is a monomial in $\mathcal{B}_{(1,1,1,2,2,5)}$.

Thus, in the matrix representation of $\nabla_{\mathcal{B}_{(1,1,1,2,2,5)}}$, there will be a -6λ as the $(2, 1)$ entry.

- (2) We repeat this process for the next monomial in the basis, $(3, 3, 3, 4, 4, 1)$. Applying the connection we get $\nabla(3, 3, 3, 4, 4, 1) = -6(4, 4, 4, 5, 5, 2)$. Since this monomial is already in \mathcal{B} we write -6 in the $(3, 2)$ position in the block matrix.
- (3) Take $\nabla(4, 4, 4, 5, 5, 2) = -6(5, 5, 5, 6, 6, 3)$. We have to do the reduction process again, represented below.



This is a bit harder to unravel than the other cases, but it works in exactly the same way. The diagram shows us that

$$\begin{aligned}
 (5, 5, 5, 6, 6, 3) &= -\frac{\lambda^2}{108}(1, 1, 1, 2, 2, 5) + \frac{17\lambda^4}{36}(3, 3, 3, 4, 4, 1) \\
 &\quad - \frac{3\lambda^5}{2}(4, 4, 4, 5, 5, 2) + \lambda^6(5, 5, 5, 6, 6, 3).
 \end{aligned}$$

And solving for $(5, 5, 5, 6, 6, 3)$ we get that

$$\begin{aligned}
 \nabla(4, 4, 4, 5, 5, 2) &= -6(5, 5, 5, 6, 6, 3) \\
 &= -\frac{\lambda^2}{18(\lambda^6 - 1)}(1, 1, 1, 2, 2, 5) + \frac{17\lambda^4}{6(\lambda^6 - 1)}(3, 3, 3, 4, 4, 1) \\
 &\quad - \frac{9\lambda^5}{\lambda^6 - 1}(4, 4, 4, 5, 5, 2).
 \end{aligned}$$

(4) Combining all of these steps, we can write $\nabla_{\mathcal{B}_{(1,1,1,2,2,5)}}$ as

$$\nabla_{\mathcal{B}_{(1,1,1,2,2,5)}} = \begin{pmatrix} 0 & 0 & -\frac{\lambda^2}{18(\lambda^6 - 1)} \\ -6\lambda & 0 & \frac{17\lambda^4}{6(\lambda^6 - 1)} \\ 0 & -6 & -\frac{9\lambda^5}{\lambda^6 - 1} \end{pmatrix}.$$

We will now describe the algorithm for finding the parameters of the differential equation for the same example monomial. The connection above gives us that differentiation with respect to λ is equivalent to the differential system

$$\frac{d}{d\lambda}y = \begin{pmatrix} 0 & -6\lambda & 0 \\ 0 & 0 & -6 \\ \lambda^2 & 17\lambda^4 & 9\lambda^5 \\ -\frac{1}{18(\lambda^6-1)} & \frac{1}{6(\lambda^6-1)} & -\frac{1}{\lambda^6-1} \end{pmatrix} y.$$

We compute the change of basis for the cyclic vector lemma, which is

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -6\lambda & 0 \\ 0 & -6 & 36\lambda \end{pmatrix}.$$

And so we have the new system

$$\frac{d}{d\lambda}y = \left(SAS^{-1} + \frac{dS}{d\lambda}S^{-1} \right) y = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{2\lambda^3}{1-\lambda^6} & \frac{10\lambda^6-2}{\lambda^2(1-\lambda^6)} & \frac{7\lambda^6+2}{\lambda(1-\lambda^6)} \end{pmatrix} y,$$

which, as we expected, is given by the companion matrix for a third order differential equation.

This system is equivalent to

$$\frac{d}{d\lambda}y = \frac{1}{\lambda} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ \frac{2\lambda^6}{1-\lambda^6} & \frac{10\lambda^6-2}{1-\lambda^6} & 2 - \frac{7\lambda^6+2}{1-\lambda^6} \end{pmatrix} y,$$

which clearly has a simple pole at $\lambda = 0$.

Now we can change variables by setting $z = \lambda^6$. The change of variables leaves us with a system

$$\frac{d}{d\lambda}y = \frac{1}{z} \begin{pmatrix} 0 & 1/6 & 0 \\ 0 & 1/6 & 1/6 \\ \frac{z}{3(1-z)} & \frac{5z-1}{3(1-z)} & \frac{5z+4}{6(1-z)} \end{pmatrix} y.$$

The residue at zero is

$$A_{-1} = \begin{pmatrix} 0 & 1/6 & 0 \\ 0 & 1/6 & 1/6 \\ 0 & -1/3 & 2/3 \end{pmatrix},$$

which has eigenvalues $0, 1/2, 1/3$.

Also, around infinity we have

$$\frac{d\tilde{y}}{d\zeta} = \frac{1}{\zeta} \begin{pmatrix} 0 & -1/6 & 0 \\ 0 & -1/6 & -1/6 \\ 1 & 5-\zeta & 5+4\zeta \\ \frac{1}{3(1-\zeta)} & \frac{5-\zeta}{3(1-\zeta)} & \frac{5+4\zeta}{6(1-\zeta)} \end{pmatrix} \tilde{y},$$

which has residue (at $\zeta = 0$) of

$$\tilde{A}_{-1} = \begin{pmatrix} 0 & -1/6 & 0 \\ 0 & -1/6 & -1/6 \\ 1/3 & 5/3 & 5/6 \end{pmatrix},$$

and thus yields the eigenvalues $1/3, 1/6, 1/6$.

We have now found the parameters of the hypergeometric differential equation associated to this connection matrix block:

$$D\left(\frac{1}{6}, \frac{1}{6}, \frac{1}{3}; \frac{1}{2}, \frac{2}{3}\right)y = 0.$$

To sum it up, the block of the matrix ∇ corresponding to the eigenspace of $(1, 1, 1, 2, 2, 5)$ gives rise to the hypergeometric differential equation which has

$${}_3F_2\left(\frac{1}{6}, \frac{1}{6}, \frac{1}{3}; \frac{1}{2}, \frac{2}{3} \middle| \lambda^6\right)$$

as its holomorphic solution around 0.

Table 1 below shows some numerical examples for $n = 6$.

TABLE 1. Parameters for $n = 6$

Monomial	α_i	β_i
$[1, 1, 1, 1, 1, 1]$	$\left[\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right]$	$\left[\frac{1}{2}, \frac{2}{3}, \frac{5}{6}, \frac{1}{3}\right]$
$[5, 3, 1, 1, 1, 1]$	$\left[\frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right]$	$\left[\frac{2}{3}, \frac{1}{3}\right]$
$[4, 4, 1, 1, 1, 1]$	$\left[\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{2}{3}\right]$	$\left[\frac{1}{3}, \frac{1}{2}, \frac{5}{6}\right]$
$[5, 2, 2, 1, 1, 1]$	$\left[\frac{1}{3}, \frac{1}{6}, \frac{1}{6}\right]$	$\left[\frac{1}{2}, \frac{2}{3}\right]$
$[4, 3, 2, 1, 1, 1]$	$\left[\frac{1}{6}, \frac{1}{6}\right]$	$\left[\frac{5}{6}\right]$
$[3, 3, 3, 1, 1, 1]$	$\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{6}, \frac{1}{6}\right]$	$\left[\frac{2}{3}, \frac{5}{6}, \frac{1}{3}\right]$
$[4, 2, 2, 2, 1, 1]$	$\left[\frac{1}{3}, \frac{1}{3}, \frac{1}{6}\right]$	$\left[\frac{1}{2}, \frac{5}{6}\right]$
$[3, 3, 2, 2, 1, 1]$	$\left[\frac{1}{2}, \frac{1}{3}, \frac{1}{6}\right]$	$\left[\frac{2}{3}, \frac{5}{6}\right]$
$[3, 2, 2, 2, 2, 1]$	$\left[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right]$	$\left[\frac{2}{3}, \frac{5}{6}\right]$
$[5, 5, 3, 3, 1, 1]$	$\left[\frac{1}{6}, \frac{1}{2}, \frac{5}{6}\right]$	$\left[\frac{1}{3}, \frac{2}{3}\right]$
$[5, 5, 4, 2, 1, 1]$	$\left[\frac{1}{6}, \frac{5}{6}\right]$	$\left[\frac{1}{2}\right]$
$[5, 4, 4, 3, 1, 1]$	$\left[\frac{1}{6}, \frac{2}{3}\right]$	$\left[\frac{1}{3}\right]$
$[5, 4, 3, 3, 2, 1]$	$\left[\frac{1}{2}\right]$	\square
$[4, 4, 4, 3, 2, 1]$	$\left[\frac{2}{3}, \frac{2}{3}\right]$	$\left[\frac{5}{6}\right]$

APPENDIX

GP SCRIPTS

Computing the connection matrix. This function counts the number of coordinates of a vector with entries bigger than or equal to d .

```
count(a)=
{
  local(t);
  t=0;
  for(k=1,length(a), if(a[k]>=length(a),t=t+1));
  t
}
```

This function (from [25]) can tell if a given element is in a vector, and gives the “position” of the element.

```
memb(g,v)=for(k=1,length(v),if(g==v[k],return(k)));0
```

The following function takes a vector a and an integer m (its coefficient) and subtracts one to all the entries and adds d to one of them until it gets to 0, a vector in the basis, or the original vector. It saves the leftovers in a vector v . It is one of the two possible reductions coming from the relations on \mathcal{W} .

```
red1(a,m)=
{
  local(j,l,b,h,t,d,s,v,u);
  h=m;
  b=0;
  t=vector(length(a));
  d=length(a);
  l=1;
  j=1;
  s=vector(length(a));
  v=vector(0);
  t=a;
  if(count(t)>0,until(count(t)==0 || t==a,
    for(k=1,length(t), if(t[k]<t[j], j=k));
    for(k=1,length(t), s[k]=t[k]-1);
    for(k=1,length(t), if(k==j, t[k]=s[k]+d, t[k]=s[k]));
    l=h*s[j]/(d*n);
    v=concat(v, [[l,s]]);
    h=h*1/n));
  [h,t]
}
```

```
red1leftovers(a,m)=
{
  local(j,l,b,h,t,d,s,v,u);
  h=m;
  b=0;
  t=vector(length(a));
  d=length(a);
```

```

l=1;
j=1;
s=vector(length(a));
v=vector(0);
t=a;
if(count(t)>0,until(count(t)==0 || t==a,
  for(k=1,length(t), if(t[k]<t[j], j=k));
  for(k=1,length(t), s[k]=t[k]-1);
  for(k=1,length(t), if(k==j, t[k]=s[k]+d, t[k]=s[k]));
  l=h*s[j]/(d*n);
  v=concat(v, [[l,s]]);
  h=h*1/n));
v
}

```

Here is the other possible reduction. This one subtracts n from one spot and adds one to everything afterwards. Saves leftovers in vector v .

```

red2(a,m)=
{
  local(j,l,b,h,t,d,s,v,u);
  h=m;
  b=0;
  d=length(a);
  l=1;
  t=vector(length(a));
  v=vector(0);
  j=1;
  s=vector(length(a));
  t=a;
  if(count(t)>0,until(count(t)==0 || t==a,
    for(k=1,length(t), if(t[j]<t[k], j=k));
    for(k=1,length(t), if(k==j, s[k]=t[k]-d, s[k]=t[k]));
    for(k=1,length(t), t[k]=s[k]+1);
    l=h*(-s[j])/d;
    v=concat(v, [[l,s]]);
    h=h*n));
  [h,t]
}

red2leftovers(a,m)=
{
  local(j,l,b,h,t,d,s,v,u);
  h=m;
  b=0;
  d=length(a);
  l=1;
  t=vector(length(a));
  v=vector(0);
  j=1;

```

```

s=vector(length(a));
t=a;
if(count(t)>0,until(count(t)==0 || t==a,
    for(k=1,length(t), if(t[j]<t[k], j=k));
    for(k=1,length(t),if(k==j,s[k]=t[k]-d,s[k]=t[k]));
    for(k=1,length(t), t[k]=s[k]+1);
    l=h*(-s[j])/d;
    v=concat(v,[[1,s]]);
    h=h*n));
    v
}

```

Now we combine these two reductions and loop until we get monomials in the basis of \mathcal{W} . The input of this function is a vector of any length and the output will be the “linear combination” of that vector in terms of the basis vectors (vectors with entries between 1 and the length).

```

reduction(a)=
{
local(d,b,c,u,v,w,uu, bb,j, t, s, g,r);
u=vector(0);
v=vector(0);
d=length(a);
j=1;
if(count(a)==d, u=[red1(a,1)]; v=red1leftovers(a,1),
    u=[red2(a,1)]; v=red2leftovers(a,1));
for(k=1, 10^d,
    if(k<=length(v),
        if(count(v[k][2])==0,
            if(v[k][1]==0, ,b=0;
                for(i=1,length(u),
                    if(v[k][2]==u[i][2],
                        u[i][1]=u[i][1]+v[k][1],
                        b=b+1));
                    if(b==length(u),u=concat(u,[v[k]]))),
            if(v[k][1]==0, ,
                uu=red2(v[k][2],v[k][1]);
                v=concat(v,red2leftovers(v[k][2],v[k][1]));
                b=0;
                for(i=1,length(u),
                    if(uu[2]==u[i][2],
                        u[i][1]=u[i][1]+uu[1],
                        b=b+1));
                    if(b==length(u),u=concat(u,[uu]))),
            break));
    b=0;
for(k=1, length(u),
    if(u[k][2]==a,
        w=vector(length(u)-1);
        for(j=1,k-1,w[j]=[u[j][1]/(1-u[k][1]),u[j][2]]);

```

```

        for(j=k,length(u)-1,
            w[j]=[u[j+1][1]/(1-u[k][1]),u[j+1][2]]),
        b=b+1));
        if(b==length(u), r=u, r=w);
    r
}

```

The next step is to write the connection matrix from this, that is, write a function that gives the derivatives of each vector in terms of the basis. In fact, there is an easy way to write the derivative of any vector using the reduction function.

```

derivative(a)=
{
    local(d,t,w);
    d=length(a);
    t=vector(d);
    for(k=1,d,t[k]=a[k]+1);
    if(count(t)==0,[[ -d,t]],
        w=reduction(t);
        for(k=1,length(w),
            w[k][1]=w[k][1]*(-d)); w)
}

```

Given a basis vector, we can find all the other basis vectors that will be a basis for the same eigenspace.

```

orbit(a)=
{
    local(l,m, c,ss);
    d=length(a);
    l=0;
    c=0;
    ss=1;
    for(k=1,d, for(t=1,d, if(k==d-a[t], l=l+1;break)));
    m=d-l;
    b=vector(m); for(k=1,m, b[k]=vector(d));
    b[1]=a;
    for(s=1,d-1,for(t=1,d, if(s==d-a[t], , c=c+1));
        if(c==d,ss=ss+1;for(t=1,d, b[ss][t]=(a[t]+s)%d);c=0);
    b;
}

```

The following gives the matrix representation of the block of the Gauss-Manin connection associated to a particular basis vector (i.e., it gives a block of the whole matrix, which is related to the eigenspace related to this basis vector).

```

connectionmatrix(a)=
{
    local(v,w,M);
    v=orbit(a);
    M=matrix(length(v),length(v));
    for(j=1,length(v),
        w=derivative(v[j]);
        for(k=1,length(w),M[memb(w[k][2],v),j]=w[k][1]));
    M=mattranspose(M);
}

```

```

M
}

```

The algorithm to find the differential equation. The following finds the derivative with respect to λ of a vector with a coefficient. It is essentially the product rule.

```

derivn(a)=
{  local(b,z, ww, vv);
   b=deriv(a[1]);
   z=derivative(a[2]);
   ww=[[b,a[2]]];
   vv=vector(length(z));
   for(k=1,length(z), vv[k]=[a[1]*z[k][1],z[k][2]]);
   for(k=1,length(vv),
       if(vv[k][2]==a[2], ww[1][1]=ww[1][1]+vv[k][1],
          ww=concat(ww,[vv[k]])));
   ww
}

```

We would like to have the derivative of a vector which is a linear combination of these monomials. This should use ideas like the function above. The first function finds the derivative of a vector (with a coefficient) in a prescribed basis determined by the orbit of b . The second does the same, but only outputs the vector of coordinates, without writing the basis down.

```

derivv(a,b)=
{  local(v,w,z);
   w=orbit(b);
   v=vector(length(w));
   for(i=1,length(w), v[i]=[0,w[i]]);
   for(k=1,length(a),
       z=derivn(a[k]);
       for(j=1,length(z),
           v[memb(z[j][2],w)][1]=v[memb(z[j][2],w)][1]+z[j][1]
       );
   );
   v
}

```

```

derivv2(a,b)=
{  local(v,w,z);
   w=orbit(b);
   v=vector(length(w));
   for(k=1,length(a),
       z=derivn(a[k]);
       for(j=1,length(z),
           v[memb(z[j][2],w)]=v[memb(z[j][2],w)]+z[j][1]
       );
   );
   v
}

```

```
}

```

We want to change basis, and we need a matrix that changes from our basis obtained by using “connection” to a basis obtained from derivatives (i.e. we use the cyclic vector theorem to write the matrix as the companion matrix to a differential equation).

```
cob(a)=
{
    local(z, vv, uu, w);
    uu=orbit(a);
    r=vector(length(uu),k,if(k==1,1));
    vv=vector(length(uu), k, [r[k],uu[k]]);
    z=[r];
    for(k=1,length(uu)-1,
        w=derivv2(vv,a);
        vv=derivv(vv,a);
        z=concat(z,[w])
    );
    S=Mat(z~);
    S
}
```

This function takes two input vectors, one is a basis and the other a vector indicating a linear combination of elements in this basis.

```
cobv(uu,r)=
{
    local(z, vv, w);
    vv=vector(length(uu), k, [r[k],uu[k]]);
    z=[r];
    for(k=1,length(uu)-1,
        w=derivv2(vv,a);
        vv=derivv(vv,a);
        z=concat(z,[w])
    );
    S=Mat(z~);
    S
}
```

The following computes what a change of basis does to the system of differential equations, where we change from a basis found by using the connection function to a basis of all the derivatives of a specific vector.

```
cobsystem(A, S)=
{
    local(dS, C);
    dS=matrix(length(A), length(A), X, Y, deriv(S[X,Y]));
    C=S*A*1/S+dS*1/S;
    C
}
```

Now, as seen in Section 5, we can change this system into an equivalent one with a simple pole at 0.

```

regform(A)=
{
local(m,Areg);
m=length(A);
Areg=A;
for(k=1,m, Areg[m,k]=Areg[m,k]*n^(m-k+1));
for(k=1,m,
  for(i=1,m, if (i==k, Areg[k,i]=Areg[k,i]+k-1)));
Areg
}

```

We need to change variables to have it in terms of z instead of λ^d .

```

varchange(A,d)=
{
for(k=1,length(A),
  for(i=1,length(A),
    A[k,i]=substpol(A[k,i],n^d,x)/d));
A
}

```

Finally we can compute the residue of the matrix at $z = 0$.

```

residuezero(A)=
{
for(k=1,length(A),
  for(i=1,length(A),
    A[k,i]=subst(A[k,i],x,0)));
A
}

```

We want a function that finds the rational roots of a polynomial with rational coefficients (because all of our characteristic polynomials are of that form and only have rational roots). We are using the rational roots theorem.

```

ratlroots(f)=
{
  local(p,q,z,vv,a,b,c,n,r,j,i);
  c=poldegree(f);
  vv=vector(c+1);
  r=vector(0);
  z=vector(0);
  for(k=1,c+1, vv[k]=polcoeff(f,k-1));
  n=denominator(vv);
  f=f*n;
  a=substpol(f,x,0);
  if(a==0, r=concat(r,0); f=f/x; a=substpol(f,x,0));
  b=pollead(f);
  p=concat(divisors(a),-divisors(a));
  q=concat(divisors(b),-divisors(b));
  for(k=1,length(p),
    for(i=1,length(q),
      if(memb(p[k]/q[i],z)==0, z=concat(z,p[k]/q[i]))));
}

```



```

    for(i=1,length(z),
        if(substpol(f,x,z[i])==0, r=concat(r,z[i]);
            f=f/(x-z[i]);i=1));
    r
}

```

We now want to combine all these steps to find the hypergeometric parameters given a vector (or monomial) in \mathcal{W} .

```

hypergcoeff(a)=
{
    local(A, B, S, Azero, Ainf,f,g, vv, uu, d, m,r,t);
    d=length(a);
    A=connectionmatrix(a);
    S=cob(a);
    A=cobsystem(A,S);
    A=regform(A);
    A=varchange(A,d);
    Azero=residuezero(A);
    f=charpoly(Azero);
    B=matrix(length(A),length(A));
    Ainf=matrix(length(A),length(A));
    for(k=1,length(A),
        for(i=1,length(A),
            B[k,i]=--substpol(A[k,i],x,1/y)));
    for(k=1,length(A),
        for(i=1,length(A),
            Ainf[k,i]=subst(B[k,i],y,0)));
    g=charpoly(Ainf);
    r=ratlroots(f);
    m=vector(length(r));
    for(k=1,length(r), m[k]=1-r[k]);
    \\ print("alphas ",ratlroots(g)," betas ",m)
    t=[ratlroots(g),m];
    t
}

```

Creating a hypergeometric table. To make a table like Table 6 we should be able to check the hypergeometric coefficients somewhat systematically. So we have to find a good way to generate basis vectors (or representatives up to permutations of the variables). We are certain that there are more efficient ways to do this, based on conversations with computational number theorists, but the focus of our project is to obtain the relationship with hypergeometric functions, which this code does.

This function (also from [25]) finds all the partitions of a number m .

```

part(m)=
{
    local(k,j,sm,sj,s, S = []);
    k = j = 1;
    sm = sj = vector(m+1);

```

```

while(k,
  s = sm[k]+j;
  if (s > m,
    until(j <= m, j = sj[k]+1; k--);
    next);

  k++; sm[k]=s; sj[k]=j;

  if (s == m,
    S = concat(S, [vector(k-1,1, sj[k-1+1])])));

S
}

```

This function uses the previous one to find the partitions of a number m of length c and into numbers that are less than c .

```

part2(m,c)=
{
  local(v,w,t);
  v=part(m);
  w=vector(0);
  for(k=1,length(v),
    if(length(v[k])==c,
      t=count(v[k]);
      if(t==0,
        w=concat(w,[v[k]]))));
  w
}

```

Now we can put this together to get representatives of the basis. We don't have strict representatives, but at least we eliminate the cases in which none of the entries are equal to one, because those obviously are in an eigenspace with a vector with entries equal to one.

```

basisreps(m)=
{
  local(v,w);
  v=vector(0);
  for(k=1,ceil((m-1)/2),
    w=part2(m*k,m);
    for(j=1, length(w),
      if(memb(1,w[j])==0, ,v=concat(v,[w[j]]))));
  v
}

```

We now want to be able to output a table with the basis vectors and the hypergeometric parameters associated to them given a number d . It should turn out to be the list of numbers in the vector that remain after canceling out with the list of numbers between 0 and d , in accordance with Katz's results [20].

```

hypergtable(d)=
{

```

```

local(v,u);
v=basisreps(d);
for(k=1,length(v), u=hypercoeff(v[k]);
print(v[k],"      ", "alphas      ", u[1],
      "      betas      ", u[2]))
}

```

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