Read directions carefully and show all your work. Partial credit will be assigned based upon the correctness, completeness, and clarity of your answers.

1. Recall that the Maclaurin series for 
   \[\sin t = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1}}{(2k+1)!}\] on \((-\infty, \infty)\).

   (a) (4 pts.) Find the first four non-zero terms of the Maclaurin series for 
   \[f(x) = 3x^2 \sin(-x)\].

   (b) (4 pts.) What is the 5th derivative at \(x = 0\) of the function \(f(x)\) in part (a)?

2. (8 pts.) Consider the initial value problem \(\frac{dy}{dx} = x^3 - y^3\) and \(y(0) = 0\). Apply Euler’s method using step size \(\Delta x = 0.2\) to estimate \(y(1)\). Round to 4 digits after the decimal. (Note: you are not required to use the table, but regardless of your method, you need to show enough work to justify your answer.)

<table>
<thead>
<tr>
<th>(x)</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y)</td>
<td>0</td>
</tr>
<tr>
<td>(\frac{dy}{dx})</td>
<td></td>
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</tbody>
</table>
3. (8 pts.) Find the interval of convergence for the series $\sum_{k=1}^{\infty} \frac{(3x - 2)^k}{k}$.

4. (8 pts.) Use the integral test to determine if $\sum_{k=1}^{\infty} \frac{1}{(\arctan k)(1 + k^2)}$ converges. If the series converges then find an exact value for both an upper and lower bound for the series. (No decimals allowed).
5. Use separation of variables to solve the following initial value problem. Be sure to use the initial condition to determine the value of any constant you introduce.

\[ \frac{dy}{dt} = \frac{te^t}{y} \text{ with } y(0) = 2 \]

6. Consider \( \sum_{k=1}^{\infty} \frac{3}{k(k+1)} \).

(a) Use partial fractions to rewrite \( a_k = \frac{3}{k(k+1)} \).

(b) Find the first 3 terms for the sequence of partial sums for \( \sum_{k=1}^{\infty} a_k \). Be sure to use your answer for \( a_k \) from part (a). Give exact values, no decimals. Do not use a calculator.

\[
S_1 = \\
S_2 = \\
S_3 = 
\]

(c) Use part (b) to find a formula for the \( n^{th} \) partial sum, \( S_n \). Then find \( \lim_{n \to \infty} S_n \).

(d) What does this tell you about \( \sum_{k=1}^{\infty} \frac{3}{k(k+1)} \)?
7. (8 pts.) A 10-inch tall cup has circular cross sections that taper (linearly) from a radius of 3 inches at the top of the cup to a radius of 2 inches at the bottom. Suppose the cup is full of strawberry milkshake that weighs 0.45 ounces/in^3. *Set up, but do not solve the integral* that represents how much work (neglecting friction) is required to suck up all of the milkshake through a straw that sticks up 3 inches above the top of the cup.
8. Consider the region in the first quadrant bounded by \( y = 0 \), \( y = \sqrt{4 - x^2} \), and the x-axis from \( x = 0 \) to \( x = 2 \).

(a) (4 pts.) \textit{Set up, but do not evaluate}, the integral representing the volume of the solid created by rotating this region around the x-axis.

(b) (4 pts.) \textit{Set up, but do not evaluate}, the integral representing the volume of the solid created by rotating this region around the line \( x = 3 \).

(c) (7 pts.) Find the arc length of \( y = \sqrt{4 - x^2} \) from \( x = 0 \) to \( x = 1 \).
   (Hint: see the last page for useful formulae.)
9. (20 pts.) True/False. Determine if each of the following statements are true or false. If a statement is false, write a sentence or two explaining why the statement is false or give an example to demonstrate why.

(a) If \( \lim_{k \to \infty} a_k = 0 \) then \( \sum_{k=1}^{\infty} a_k \) converges.

(b) The improper integral \( \int_{0}^{1} \frac{1}{x^p} \, dx \) diverges for \( p \geq 1 \).

(c) The geometric series \( 1 + \frac{\pi}{e} + \left(\frac{\pi}{e}\right)^2 + \left(\frac{\pi}{e}\right)^3 + \left(\frac{\pi}{e}\right)^4 + \cdots \) converges to \( \frac{1}{1 - \frac{\pi}{e}} = \frac{e}{e - \pi} \).

(d) If the terms of a series alternate in sign, decrease in magnitude, and approach zero, then the series converges.

(e) The ratio test can be used to show that the alternating harmonic series \( \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \) converges.

(f) If \( 0 \leq f(x) \leq g(x) \) and \( \int_{1}^{\infty} g(x) \, dx \) diverges, then \( \int_{1}^{\infty} f(x) \, dx \) diverges.

(g) If \( 0 \leq a_k \leq \frac{1}{k^{3/2}} \) for \( k \geq 1 \), then \( \sum_{k=1}^{\infty} a_k \) converges.

(h) If \( f \) is increasing and concave up, then \( L_{20} \leq M_{20} \leq \int_{a}^{b} f(x) \, dx \leq T_{20} \leq R_{20} \).

(i) The series \( 1 - \frac{1}{2!} \left(\frac{\pi}{4}\right)^2 + \frac{1}{4!} \left(\frac{\pi}{4}\right)^4 - \frac{1}{6!} \left(\frac{\pi}{4}\right)^6 + \frac{1}{8!} \left(\frac{\pi}{4}\right)^8 - \cdots \) converges to 1.

(j) Given a first-order differential equation \( \frac{dy}{dx} = f(x,y) \) with an initial condition \( y(x_0) = y_0 \), Euler’s method produces an exact solution to the initial value problem in the form an equation for the solution curve.
10. (7 pts.) Suppose the number of hours before a transistor fails is a random variable $T$ whose probability density function is given by

$$f(t) = \begin{cases} 
kte^{-t^2} & \text{for } t > 0 \\
0 & \text{for } t \leq 0 
\end{cases}$$

Find the value of $k$ that ensures $f$ is a probability density function.

**EXTRA CREDIT**

The **Root Test** is another test that can be used to determine the convergence of a series. Let $\sum_{n=1}^{\infty} a_n$ be a series with $a_n \geq 0$ for $n \geq N$, and suppose that $\lim_{n \to \infty} \sqrt[n]{a_n} = L$. Then,

- $\sum_{n=1}^{\infty} a_n$ converges if $L < 1$,
- $\sum_{n=1}^{\infty} a_n$ diverges if $L > 1$ or if $L$ is infinite,
- the test is inconclusive if $L = 1$.

Use the Root test to determine if $\sum_{n=1}^{\infty} \frac{(\ln n)^n}{1000^n}$ converges or diverges.
Formulae you may find useful.

- \[ \int \sec \theta \, d\theta = \ln |\sec \theta + \tan \theta| + C \]
- \[ \int \csc \theta \, d\theta = \ln |\csc \theta - \cot \theta| + C \]
- \[ \int \cot \theta \, d\theta = -\ln |\csc \theta| + C \]
- \[ \int_a^b \sqrt{1 + (f'(x))^2} \, dx \]
- \[ P_n(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k \]
- \[ |f(x) - P_n(x)| \leq \frac{K_{n+1}}{(n+1)!} |x - x_0|^{n+1} \]
- \[ \sum_{k=1}^{\infty} \frac{1}{k^p} \text{ converges for } p > 1 \]

- Log Properties
  - \( \ln(xy) = \ln x + \ln y \)
  - \( \ln \frac{x}{y} = \ln x - \ln y \)
  - \( \ln x^y = y \ln x \)

- Trigonometric Substitution

<table>
<thead>
<tr>
<th>General Form</th>
<th>( a^2 + x^2 )</th>
<th>( a^2 - x^2 )</th>
<th>( x^2 - a^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Substitution</td>
<td>( x = a \tan \theta )</td>
<td>( x = a \sin \theta )</td>
<td>( x = a \sec \theta )</td>
</tr>
</tbody>
</table>

- Trigonometric Identities

- \( \sin^2 \theta + \cos^2 \theta = 1 \)
- \( \sin^2 \theta = \frac{1}{2}(1 - \cos(2\theta)) \)
- \( \cos^2 \theta = \frac{1}{2}(1 + \cos(2\theta)) \)
- \( \sec^2 \theta = 1 + \tan^2 \theta \)
- \( \sin(2\theta) = 2 \sin \theta \cos \theta \)
- \( \cos(2\theta) = \cos^2 \theta - \sin^2 \theta \)

- Error bounds for left, right, trapezoid, and midpoint approximating sums

- \( |I - L_n| \leq \frac{K_1(b-a)^2}{2n} \)
- \( |I - R_n| \leq \frac{K_1(b-a)^2}{2n} \)
- \( |I - T_n| \leq \frac{K_2(b-a)^3}{12n^2} \)
- \( |I - M_n| \leq \frac{K_2(b-a)^3}{24n^2} \)