Final Exam

Show all your work to receive full credit for a problem.

1. Let

\[ f(x, y) = \frac{x + y^2}{2x + y}. \]

(a) Find the domain of \( f \).

(b) Show that \((0, 0)\) is a non-removable discontinuity.
2. Let \( g(x, y, z) = x^2 + y^2 + z^2 \).

(a) Describe the level surfaces of \( g \). (You can do this in many ways: verbally, through
equations, or through pictures, for example).

(b) Find a parametrization for the level surface corresponding to \( c = 9 \).

(c) Find the equation for the plane tangent to the surface corresponding to \( c = 9 \) at
the point \((1, 5, -1)\).
3. Let $\mathbf{F}$ be the vector field defined by $\mathbf{F} = x\mathbf{i} - y\mathbf{j}$. Let $C$ be the path that is made up of two line segments, the first goes from $(0, 2)$ to $(0, 0)$ and the second from $(0, 0)$ to $(2, 2)$.

(a) Draw six vectors in the vector field $\mathbf{F}$ in the coordinate axes given below.

(b) Evaluate

$$\int_C \mathbf{F} \cdot d\mathbf{x}$$
4. Let $f(x, y)$ be a function that is differentiable everywhere. At a certain point $P$ in the $xy$-plane, the directional derivative of $f$ in the direction of $i - j$ is $\sqrt{2}$ and the directional derivative of $f$ in the direction of $i + j$ is $3\sqrt{2}$.

(a) What is the gradient of $f$ at $P$?

(b) What is the maximum directional derivative of $f$ at $P$?

5. Let $f$ be a differentiable function such that, at the critical point $a$, the Hessian is

$$Hf(a) = \begin{pmatrix} -6 & 0 & 1 \\ 0 & -3 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Determine whether $a$ is a local maximum, a local minimum, or a saddle point.
6. Let $S$ be the solid bounded above by $z = e^{1-x^2-y^2}$, and below by $z = 1$, and let $\mathbf{G} = x\mathbf{i} + y\mathbf{j} + (2 - 2z)\mathbf{k}$.

(a) What is $\partial S$? (You don’t need to draw it).

(b) Assuming that $\partial S$ is oriented by an outward pointing normal, use the divergence theorem to calculate

$$\int\int_{\partial S} \mathbf{G} \cdot \mathbf{n} \, d\sigma$$

(c) Give a physical interpretation of the result you obtained in part (b).
7. Let \( f(x, y) = x^2 - 3y^2 \) and \( g(s, t) = (st, s + t^2) \).

(a) Calculate \( D(f \circ g)(1, -1) \)

(b) Suppose \( a \) is a point very close to \((1, -1)\). Explain how you would use part (a) to find an approximate value for \((f \circ g)(a)\).
8. Let $M$ be the surface defined by $x^2 + y^2 + 5z = 1$, $z \geq 0$, oriented by upward normal. Let $F$ be the vector field $F = xz\mathbf{i} + yz\mathbf{j} + (x^2 + y^2)\mathbf{k}$.

(a) Describe $\partial M$, the boundary of $M$.

(b) Use Stokes’s Theorem to show that $\oint_{\partial M} F \cdot d\mathbf{x} = 0$.

(c) Does your result for part (b) imply that $F$ is path independent? Explain why or why not.
9. Use triple integrals to verify that the volume of a ball of radius $r$ is $\frac{4\pi r^3}{3}$. 