Math 106: Review for Final Exam, Part II - SOLUTIONS

1. Use a second-degree Taylor polynomial to estimate $\sqrt[3]{28}$.
   We choose $f(x) = \sqrt[3]{x}$ and $x_0 = 27$ because 27 is the perfect cube closest to 28.
   
   
   $f(x) = x^{1/3}$
   $f(27) = 3$
   $f'(x) = \frac{1}{3}x^{-2/3}$
   $f'(27) = \frac{1}{3 \cdot 27^{2/3}} = \frac{1}{27}$
   $f''(x) = -\frac{2}{9}x^{-5/3}$
   $f''(27) = -\frac{2}{9 \cdot 27^{5/3}} = -\frac{2}{2187}$
   
   Now plug in to the Taylor polynomial formula with $x_0 = 27$.

   $P_2(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 = 3 + \frac{1}{27} (x - 27) - \frac{1}{2187} (x - 27)^2$

   Finally, evaluate at $x = 28$.

   $\sqrt[3]{28} \approx P_2(28) = 3 + \frac{1}{27}(28 - 27) - \frac{1}{2187}(28 - 27)^2 = \frac{6641}{2187} \approx 3.0365797$

2. What is the largest possible error that could have occurred in your previous estimate?
   We know that $|f(x) - P_n(x)| \leq \frac{K_{n+1}}{(n+1)!}|x - x_0|^{n+1}$.

   In this case, $n = 2$, $x_0 = 27$, and $x = 28$.

   $K_3 = \text{max of } |f'''(x)| \text{ on } [27, 28] = \text{max of } \left| \frac{10}{27x^{5/3}} \right| \text{ on } [27, 28] = \frac{10}{27 \cdot 27^{8/3}} = \frac{10}{177147}$

   Putting this all together, we have $|f(x) - P_2(x)| \leq \frac{10}{3 \cdot 27^3} (28 - 27)^3 = \frac{5}{531441} \approx 0.0000094$.

3. Use a comparison to show whether each of the following converges or diverges. If an integral converges, give a good upper bound for its value.

   (a) $\int_1^\infty \frac{7 + 5 \sin x}{x^2} \, dx$

   For all $x \geq 1$, we have $0 \leq \frac{7 + 5 \sin x}{x^2} \leq \frac{7 + 5(1)}{x^2} = \frac{12}{x^2}$ because the maximum of $\sin x$ is 1.

   $12 \int_1^\infty \frac{dx}{x^2} = 12 \lim_{t \to \infty} \int_1^t \frac{dx}{x^2}$

   $= 12 \lim_{t \to \infty} \left[ -\frac{1}{x} \right]_1^t$

   $= 12 \lim_{t \to \infty} \left[ -\frac{1}{t} - \frac{-1}{1} \right]$

   $= 12[0 - (-1)]$

   $= 12$

   Therefore, the original integral in question must converge to a value less than 12.

   (b) $\int_1^\infty \frac{1 + 3x^2 + 2x^3}{\sqrt{10x^{12} + 17x^{10}}} \, dx$

   For $x \geq 1$, we have $\frac{1 + 3x^2 + 2x^3}{\sqrt{10x^{12} + 17x^{10}}} \geq \frac{2x^3}{\sqrt{10x^{12} + 17x^{10}}} \geq 0$. (We've made the numerator smaller and the denominator larger, so the new fraction is smaller.)
But \( \frac{2x^3}{\sqrt{10x^2 + 17x^2}} = \frac{2x^3}{\sqrt{27x^2}} = \frac{2x^3}{3x^2} = \frac{2}{3} x \) and we know that \( \frac{2}{3} \int_1^\infty \frac{dx}{x} \) diverges (compute for yourself or notice that \( p = 1 \)).

Therefore the original integral must also diverge.

4. Decide if each of the following sequences \( \{a_k\}_{k=1}^\infty \) converges or diverges. If a sequence converges, compute its limit.

(a) \( a_k = 3 + \frac{1}{10^k} \)

Terms are 3.1, 3.01, 3.001, 3.0001, ....

\[ \lim_{k \to \infty} \left( 3 + \frac{1}{10^k} \right) = 3, \text{ so the sequence converges to } 3. \]

(b) \( a_k = (-1)^k \)

Terms are \(-1, 1, -1, 1, ....\)

\[ \lim_{k \to \infty} (-1)^k \text{ doesn't exist, so the sequence diverges}. \]

(c) \( a_k = \frac{3 + 5k}{7 + 2k} \)

Terms are \(8/9, 13/11, 18/13, 23/15, ....\)

\[ \lim_{k \to \infty} \frac{3 + 5k}{7 + 2k} = \frac{5}{2} \text{ (by L'Hopital's Rule or by inspection), so the sequence converges to } \frac{5}{2}. \]

5. Decide if each of the following series converges or diverges. If a series converges, find its value.

(a) \( 3.1 + 3.01 + 3.001 + 3.0001 + ... \)

\[ \lim_{k \to \infty} a_k = 3 \neq 0, \text{ so the series diverges by the nth Term Test. (We keep adding 3's forever.)} \]

[Compare this with the first sequence of the previous problem.]

(b) \( 1 + 1/2 + 1/3 + 1/4 + ... \)

This is the famous Harmonic Series, which diverges even though the terms do approach 0. We can use the Integral Test: \( \int_1^\infty \frac{1}{x} dx \) diverges, which means that \( \sum_{k=1}^\infty \frac{1}{k} \) must diverge too.

(c) \( 5 - 5/3 + 5/9 - 5/27 + ... \)

This is a geometric series with \( r = -\frac{1}{3}, \) so it converges to \( \frac{a}{1-r} = \frac{5}{1-(-1/3)} = \frac{15}{4}. \)

6. Decide if each of the following series converges or diverges. If a series converges, find upper and lower bounds for its value.

(a) \( \sum_{k=1}^\infty \frac{(-1)^k}{\sqrt{k+1}} \) [Alternating Series Test]

The terms of this series alternate in sign.

And, \( \frac{1}{\sqrt{2}} \geq \frac{1}{\sqrt{3}} \geq \frac{1}{\sqrt{4}} \geq ... \geq 0. \) (Or, more formally, \( k < k + 1 \Rightarrow (k + 1) < (k + 1) + 1 \Rightarrow \sqrt{(k + 1)} < \sqrt{(k + 1) + 1} \Rightarrow \frac{1}{\sqrt{(k + 1)}} < \frac{1}{\sqrt{(k + 1) + 1}} \) so the terms are decreasing in size.)

And, \( \lim_{k \to \infty} \frac{1}{\sqrt{k+1}} = 0. \)

Therefore, by the Alternating Series Test, the series must converge.

We know that any two consecutive partial sums will provide upper and lower bounds:

lower bound = \( S_1 = -\frac{1}{\sqrt{2}} \)
upper bound = \( S_2 = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} \)

[To get better bounds, use later partial sums, such as \( S_5 \) and \( S_6 \).]
(b) $\sum_{k=1}^{\infty} \frac{(2k)!}{3^k(k!)^2}$

$$\lim_{k \to \infty} \frac{a_{k+1}}{a_k} = \frac{\frac{[2(k+1)]!}{3^{k+1}[(k+1)!]^2}}{\frac{2k!}{3^k(k!)^2}} = \frac{(2k+2)! 3^k (k!)^2}{(2k)! 3^{k+1} [(k+1)!]^2}$$

$$= \lim_{k \to \infty} \frac{(2k+2)(2k+1) 1}{3 (k+1)^2}$$

$$= \frac{4k^2 + 6k + 2}{3k^2 + 6k + 3}$$

Use L'Hopital or divide each term by $k^2$.  

Since the limit of the ratio is greater than 1, the series diverges.

(c) $\sum_{k=1}^{\infty} \left( \frac{1}{100} + \frac{1}{k^5} \right)$

$$\lim_{k \to \infty} \left( \frac{1}{100} + \frac{1}{k^5} \right) = \frac{1}{100} \neq 0,$$

so, by the nth Term Test, the series diverges.

(d) $\sum_{k=1}^{\infty} \frac{\sqrt{9k^8 + 5k^6}}{12k^5 + 3}$

[Comparison Test]

For $k \geq 1$, we have $\frac{\sqrt{9k^8 + 5k^6}}{12k^5 + 3} > \frac{\sqrt{9k^8}}{12k^5 + 3k^5} > 0$.

But, $\frac{\sqrt{9k^8}}{12k^5 + 3k^5} = \frac{3k^4}{15k^5} = \frac{1}{5k}$ and we know that $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges (use Integral Test or note that $p = 1$).

Therefore, the original series, which has larger terms, must diverge also.

(e) $\sum_{k=2}^{\infty} \frac{1}{k(ln(k))^2}$

[Integral Test]

$$\int_{2}^{\infty} \frac{1}{x(ln x)^2} \, dx = \lim_{t \to \infty} \int_{x=2}^{x=t} \frac{1}{x(ln x)^2} \, dx$$

$$= \lim_{t \to \infty} \int_{x=2}^{x=t} \frac{1}{u^2} \, du$$

Substitute $u = \ln x$, so $du = \frac{dx}{x}$.

$$= \lim_{t \to \infty} \frac{u^{-1}}{x=2}^{x=t}$$

$$= \lim_{t \to \infty} \frac{-1}{\ln t}^{x=t}$$

$$= \lim_{t \to \infty} \frac{-1 - \frac{1}{\ln 2}}{\ln t}$$

$$= \frac{-1}{\ln 2}$$

$$= \frac{1}{\ln 2}$$

The integral converges, so the series must converge too.
Further, we know that \( \int_{2}^{\infty} \frac{1}{x \ln x} \, dx \leq \sum_{k=2}^{\infty} \frac{1}{k \ln(k)} \leq a_2 + \int_{2}^{\infty} \frac{1}{x \ln x} \, dx. \)

Therefore, our lower bound is \( \int_{2}^{\infty} \frac{1}{x \ln x} \, dx = \frac{1}{\ln 2}. \)

And our upper bound is \( a_2 + \int_{2}^{\infty} \frac{1}{x \ln x} \, dx = \frac{1}{2(\ln 2)^2} + \frac{1}{\ln 2}. \)

7. Does the first series from the previous problem converge absolutely or conditionally?

\[ \sum_{k=1}^{\infty} \left| \frac{(-1)^k}{\sqrt{k} + 1} \right| = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k} + 1}, \]

which diverges by the Integral Test (check for yourself).

Therefore, the first series from the previous problem converges conditionally.

8. Compute the radius and interval (including endpoints) of convergence for \( \sum_{k=1}^{\infty} \frac{(x + 3)^k}{k \cdot 5^k}. \)

\[ \lim_{k \to \infty} \frac{|(x + 3)^{k+1}|}{|(x + 3)^k|} = \lim_{k \to \infty} \frac{|(x + 3)^{k+1} / (x + 3)^k|}{(k+1)/k} = \left| \frac{x + 3}{5} \right| \]

Use L'Hopital on the middle fraction.

So, we are guaranteed convergence when \( \left| \frac{x + 3}{5} \right| < 1. \) But this is equivalent to the following.

\[ -1 < \frac{x + 3}{5} < 1 \]
\[ -5 < x + 3 < 5 \]
\[ -8 < x < 2 \]

To check convergence at the endpoints (where the Ratio Test is inconclusive), we plug in to the series itself.

At \( x = 2, \) we have \( \sum_{k=1}^{\infty} \frac{(2 + 3)^k}{k \cdot 5^k} = \sum_{k=1}^{\infty} \frac{5^k}{k \cdot 5^k} = \sum_{k=1}^{\infty} \frac{1}{k}, \) which is the Harmonic Series and thus diverges.

At \( x = -8, \) we have \( \sum_{k=1}^{\infty} \frac{(-8 + 3)^k}{k \cdot 5^k} = \sum_{k=1}^{\infty} \frac{(-5)^k}{k \cdot 5^k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k}, \) which converges by the Alternating Series Test.

Thus, the interval of convergence is \(-8 \leq x < 2\) and the radius of convergence is 5.

9. Evaluate the following exactly.

(a) \( 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \ldots \)

We recall that \( e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots. \)

The series in question is the series for \( e^x \) with \( x = -1; \) therefore, it converges to \( e^{-1}. \)

(b) \( \frac{8}{3} - \frac{8}{9} + \frac{8}{27} - \frac{8}{81} + \ldots \)

We recognize this as a geometric series with \( r = -1/3 \) and \( a = 8/3; \) therefore, it converges to \( \frac{a}{1 - r} = \frac{8/3}{1 - (-1/3)} = 2. \)
(c) \[1 - \frac{\pi^2}{2} + \frac{\pi^4}{24} - \frac{\pi^6}{720} + \ldots\]

We recall that \(\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \ldots\).

The series in question is the series for \(\cos x\) with \(x = \pi\); therefore, it converges to \(\cos \pi\), which is \(-1\).

10. (a) **Write the complete series equal to \(\int_0^1 e^{-x^2} \, dx\) and show that it converges.**

We know \(e^w = 1 + \frac{w}{1!} + \frac{w^2}{2!} + \frac{w^3}{3!} + \ldots\), so by substitution we obtain the following.

\[e^{-x^2} = 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \ldots = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \ldots\]

Thus, \(\int_0^1 e^{-x^2} \, dx = \int_0^1 \left[1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \ldots\right] \, dx\)

\[= \left[x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \ldots\right]_0^1\]

\[= 1 - \frac{1^3}{3} + \frac{1^5}{5 \cdot 2!} - \frac{1^7}{7 \cdot 3!} + \ldots\]

\[= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)k!}\]

The terms of this series alternate in sign.

And, \(1 \geq \frac{1^3}{3} \geq \frac{1^5}{5 \cdot 2!} \geq \frac{1^7}{7 \cdot 3!} \geq \ldots \geq 0\).

And, \(\lim_{k \to \infty} \frac{1}{(2k+1)k!} = 0\).

Therefore, by the Alternating Series Test, the series must converge.

(b) **If \(f(x) = e^{-x^2}\), what is \(f^{(400)}(0)\)? What is \(f^{(401)}(0)\)?**

In the previous part, we found the following.

\[e^{-x^2} = 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \ldots = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \ldots\]

We know that \(f^{(400)}(0)\) will appear in the Taylor series for \(f(x)\) in the coefficient of the \(x^{400}\) term, which is \(\frac{f^{(400)}(0)}{400!} x^{400}\).

From above, we see that term is \(\frac{x^{400}}{200!}\).

Setting them equal gives \(\frac{f^{(400)}(0)}{400!} x^{400} = \frac{x^{400}}{200!}\), which means that \(f^{(400)}(0) = \frac{200!}{200!}\).

[Note that this does not simplify to 200!. Rather, \(\frac{400!}{200!} = 400 \cdot 399 \cdot \ldots \cdot 202 \cdot 201\).]

We know that \(f^{(401)}(0)\) will appear in the Taylor series for \(f(x)\) in the coefficient of the \(x^{401}\) term, which is \(\frac{f^{(401)}(0)}{401!} x^{401}\).

However, the Taylor series above for \(f(x)\) has no terms with odd powers of \(x\), meaning that the coefficients of all those terms must be 0.

Therefore, we know that \(f^{(401)}(0) = 0\).