Math 106: Review for Final Exam, Part I - SOLUTIONS

1. Find the following. [See Review for Exam II for integration tips and strategies.]

(a) Let \( u = x^3 \), so \( du = 3x^2 \, dx \) and \( du/3 = x^2 \, dx \).

\[
\int 12x^2 \cos(x^3) \, dx = 12 \int \cos(x^3) x^2 \, dx \\
= 12 \int \cos(u) \frac{du}{3} \\
= 4 \sin(u) + C \\
= 4 \sin(x^3) + C
\]

(b) We’ll use integration by parts: \( u = x \Rightarrow du = dx \) and \( dv = e^{-3x} \Rightarrow v = \frac{e^{-3x}}{-3} \).

\[
\int_0^\infty xe^{-3x} \, dx = \lim_{t \to \infty} \int_0^t xe^{-3x} \, dx \\
= \lim_{t \to \infty} \left[ \frac{xe^{-3x}}{-3} \bigg|_0^t - \int_0^t \frac{e^{-3x}}{-3} \, dx \right] \\
= \lim_{t \to \infty} \left[ \frac{-xe^{-3x}}{3} - \frac{1}{9} \right]_0^t \\
= \lim_{t \to \infty} \left[ \frac{-t}{3e^{3t}} - \frac{1}{9e^{3t}} \right] - \left[ 0 - \frac{1}{9} \right] \\
= 0 - (0 - 1/9) \\
= 1/9
\]

So, the integral converges (to this value).

(c) This integral is improper at \( x = 4 \) because the integrand has a vertical asymptote there, so we split into two integrals.

\[
\int_0^6 \frac{dx}{(x - 4)^2} = \int_0^4 \frac{dx}{(x - 4)^2} + \int_4^6 \frac{dx}{(x - 4)^2} \\
= \lim_{a \to 4^-} \int_0^a \frac{dx}{(x - 4)^2} + \lim_{b \to 4^+} \int_b^6 \frac{dx}{(x - 4)^2} \\
= \lim_{a \to 4^-} \left[ \frac{-1}{(x - 4)} \right]_0^a + \lim_{b \to 4^+} \left[ \frac{-1}{(x - 4)} \right]_b^6 \\
= \lim_{a \to 4^-} \left[ \frac{-1}{(a - 4)} - \frac{-1}{(0 - 4)} \right] + \lim_{b \to 4^+} \left[ \frac{-1}{(6 - 4)} - \frac{-1}{(b - 4)} \right] \\
\]

Since \( \lim_{a \to 4^-} \frac{-1}{(a - 4)} = \infty \) and \( \lim_{b \to 4^+} \frac{-1}{(b - 4)} = \infty \), this integral diverges (to \( \infty \)).

(d) Partial Fractions:

Write \( \frac{3x^2 + 2x - 5}{(x^2 + 1)(x - 4)} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x - 4} \). Now multiply both sides by \( (x^2 + 1)(x - 4) \) to get

\[
3x^2 + 2x - 5 = (Ax + B)(x - 4) + C(x^2 + 1).
\]

Let \( x = 4 \). Then \( 51 = C(17) \), so \( C = 3 \).
Let \( x = 0 \). Then \( -5 = B(-4) + 3(1) \), so \( B = 2 \).
Let \( x = 1 \). Then \( 0 = (A(1) + 2)(-3) + 3(2) \), so \( A = 0 \).

\[
\int \frac{3x^2 + 2x - 5}{(x^2 + 1)(x - 4)} \, dx = \int \left[ \frac{2}{x^2 + 1} + \frac{3}{x - 4} \right] \, dx = 2 \arctan x + 3 \ln | x - 4 | + C
\]

(e) Let \( u = \sec x \), so \( du = \sec x \tan x \, dx \).

New limits: \( x = 0 \Rightarrow u = \sec 0 = 1 \) and \( x = \pi/3 \Rightarrow u = \sec(\pi/3) = 2 \).

\[
\int_0^{\pi/3} \tan^3 x \sec^6 x \, dx = \int_0^{\pi/3} \tan^3 x \sec^4 x \sec x \tan x \, dx \quad \text{Break off a } \sec x \tan x.
\]

\[
= \int_0^{\pi/3} (\sec^2 x - 1) \sec^4 x \sec x \tan x \, dx \quad \text{Use } \tan^2 x = \sec^2 x - 1.
\]

\[
= \int_1^2 (u^2 - 1)u^4 \, du \quad \text{Change the limits. See above.}
\]

\[
= \int_1^2 (u^6 - u^4) \, du \quad \text{Use } \sec^2 t = 1 + \tan^2 t \text{ or table #42.}
\]

\[
= \left[ \frac{u^7}{7} - \frac{u^5}{5} \right]_1 = \frac{2^7}{7} - \frac{2^5}{5} - \left( \frac{1^7}{7} - \frac{1^5}{5} \right) = \frac{418}{35} \quad \text{This is about 11.943.}
\]

(f) Let \( x = 5 \sin t \), so \( dx = 5 \cos t \, dt \).

\[
x^2 + y^2 = 5^2 \Rightarrow y = \sqrt{25 - x^2} \quad \sin t = \frac{\text{opp}}{\text{hyp}} = \frac{x}{5} \Rightarrow t = \arcsin(x/5)
\]

\[
\cos t = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{25 - x^2}}{5} \Rightarrow 5 \cos t = \sqrt{25 - x^2}
\]

\[
\int \sqrt{25 - x^2} \, dx = \int 5 \cos t \cdot 5 \cos t \, dt \quad \text{Use } dx \text{ and } \cos t \text{ from above.}
\]

\[
= \int 25 \cos^2 t \, dt
\]

\[
= 25 \left[ \frac{1}{2} + \frac{\cos(2t)}{2} \right] \, dt \quad \text{Use } \cos^2 t = \frac{1}{2} + \frac{\cos(2t)}{2} \text{ or table #42.}
\]

\[
= 25 \left[ \frac{t}{2} + \frac{\sin(2t)}{4} \right] + C \quad \text{Let } u = 2t \text{ to integrate } \cos(2t).
\]

\[
= 25 \left[ \frac{\arcsin(x/5)}{2} + \frac{2 \sin t \cos t}{4} \right] + C \quad \text{Use } \sin(2t) = \sin t \cos t \text{ and } x \text{ from above.}
\]

\[
= 25 \left[ \frac{\arcsin(x/5)}{2} + \frac{2 \cdot \frac{x}{5} \cdot \sqrt{25 - x^2}}{4} \right] + C \quad \text{Use } \sin t \text{ and } \cos t \text{ from above.}
\]

\[
= 25 \left[ \frac{\arcsin(x/5)}{2} + \frac{x \sqrt{25 - x^2}}{50} \right] + C
\]
2. Find the best possible left, right, midpoint and trapezoidal approximations to $\int_{-2}^{0} f(x) \, dx$ given the data in the table below.

<table>
<thead>
<tr>
<th>$x$</th>
<th>-2</th>
<th>-1.5</th>
<th>-1</th>
<th>-0.5</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>2</td>
<td>3</td>
<td>6</td>
<td>10</td>
<td>11</td>
</tr>
</tbody>
</table>

$L_4 = (2 + 3 + 6 + 10)(0.5) = 10.5$ \quad $R_4 = (3 + 6 + 10 + 11)(0.5) = 15$ \quad $T_4 = 0.5(L_4 + R_4) = 12.75$

We cannot compute $M_4$, which would require the values of $f$ at $x = -1.75, -1.25, -0.75,$ and $-0.25$. Instead, we find $M_2 : M_2 = (3 + 10)(1) = 13$.

3. If you use numerical integration to estimate $\int_{a}^{b} \ln x \, dx$, how would the following be ordered from least to greatest? $L_{100}, R_{100}, M_{100}, T_{100}, \int_{a}^{b} \ln x \, dx$.

The integrand is increasing and concave down, so we have $L_{100} < T_{100} < \int_{a}^{b} \ln x \, dx < M_{100} < R_{100}$.

4. Find bounds for each of the following errors if $I = \int_{0}^{2} e^{-5x} \, dx$.

(a) $|I - R_{100}| \leq \frac{K_1(b-a)^2}{2n} = \frac{5(2-0)^2}{2(100)} = \frac{1}{10}$

$K_1 = \max |f'(x)|$ on $[0, 2] = \max 5e^{-5x}$ on $[0, 2] = 5$ (occurs at $x = 0$)

(b) $|I - T_{100}| \leq \frac{K_2(b-a)^3}{12n^2} = \frac{25(2-0)^3}{12(100)^2} = \frac{1}{600}$

$K_2 = \max |f''(x)|$ on $[0, 2] = \max 25e^{-5x}$ on $[0, 2] = 25$ (occurs at $x = 0$)

(c) $|I - M_{100}| \leq \frac{K_2(b-a)^3}{24n^2} = \frac{25(2-0)^3}{24(100)^2} = \frac{1}{1200}$

$K_2$ same as in previous part

5. If $I = \int_{0}^{2} e^{-5x} \, dx$, how many subdivisions are required to obtain a midpoint sum approximation with error of at most $1/1,000,000$?

From part (c) above, we know that $|I - M_n| \leq \frac{K_2(b-a)}{24n^2} = \frac{25(2-0)}{24n^2} = \frac{25}{3n^2}$.

Thus, we want $\frac{25}{3n^2} \leq \frac{1}{1,000,000}$, which is equivalent to $\frac{25,000,000}{3} \leq \frac{n^2}{1}$.

Taking the square root of each side results in $\sqrt{25,000,000} \leq n$.

Since $\sqrt{25,000,000} = 2886.751\ldots$, we must at least 2887 subdivisions.
6. Use Euler’s Method with 3 steps to estimate \( y(\frac{3}{4}) \) if \( \frac{dy}{dx} = y - 3 \) and \( y(0) = 1 \).

\[
\begin{array}{|c|c|c|}
\hline
 x & y & \frac{dy}{dx} \cdot \Delta x = \Delta y \\
\hline
 0 & 1 & (-2)(0.25) = -0.5 \\
 0.25 & 0.5 & (-2.5)(0.25) = -0.625 \\
 0.5 & -0.125 & (-3.125)(0.25) = -0.78125 \\
 0.75 & -0.90625 & \\
\hline
\end{array}
\]

7. Write an integral equal to the area between \( y = 2x + 3 \) and \( y = x^2 + 7x - 3 \).

First, find where the curves intersect.

\[
x^2 + 7x - 3 = 2x + 3 \\
x^2 + 5x - 6 = 0 \\
(x + 6)(x - 1) = 0 \\
\Rightarrow x = -6, x = 1
\]

Between \( x = -6 \) and \( x = 1 \), \( y = 2x + 3 \) is above \( y = x^2 + 7x - 3 \). (Plug in \( x = 0 \) to check.) So, the area between them is \( \int_{-6}^{1} [(2x + 3) - (x^2 + 7x - 3)] \, dx \). [This equals 343/6.]

8. Compute the arc length of \( y = \sqrt{1 - x^2} \) from \( x = 0 \) to \( x = 1/2 \).

First, we find \( f'(x) = \frac{1}{2}(1 - x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{1 - x^2}} \).

Thus, \( [f'(x)]^2 = \frac{x^2}{1 - x^2} \).

\[
\int_{0}^{1/2} \sqrt{1 + [f'(x)]^2} \, dx = \int_{0}^{1/2} \sqrt{1 + \frac{x^2}{1 - x^2}} \, dx \\
= \int_{0}^{1/2} \sqrt{\frac{1 - x^2 + x^2}{1 - x^2}} \, dx \\
= \int_{0}^{1/2} \sqrt{\frac{1}{1 - x^2}} \, dx \\
= \int_{0}^{1/2} \frac{1}{\sqrt{1 - x^2}} \, dx \\
= \arcsin x \bigg|_{0}^{1/2} \\
= \arcsin(1/2) - \arcsin(0) \\
= \pi/6 - 0 \\
= \pi/6
\]

9. Consider the region bounded by \( y = 0 \), \( x = 2 \), and \( y = x^2 \). Write an integral equal to the volume of the object created when the region is revolved about

(a) the \( z \)-axis

Slice vertically into disks.
volume of slice \( \approx \pi r^2 \Delta x \)
\[ = \pi y^2 \Delta x \]
\[ = \pi (x^2)^2 \Delta x \]
\[ = \pi x^4 \Delta x \]
total volume \( = \pi \int_0^2 x^4 \, dx \)

(b) \textit{the line } x = 5 \textit{ Slice horizontally into washers.}

\[
\text{volume of slice } \approx \pi R^2 \Delta y - \pi r^2 \Delta y \\
= \pi (5 - x)^2 \Delta y - \pi (3)^2 \Delta y \\
= \pi [(5 - \sqrt{y})^2 - 3^2] \Delta y \\
total \text{ volume } = \pi \int_0^4 [(5 - \sqrt{y})^2 - 3^2] \, dy
\]

10. A spherical tank of radius 8 feet is buried 5 feet below ground and filled to a height of 11 feet with gasoline (42 pounds per cubic foot). Write an integral equal to the work done in pumping all the gasoline to ground level.

\[
\text{volume of slice } \approx \pi r^2 \Delta h = \pi (16h - h^2) \Delta h \\
\text{weight of slice } \approx 42\pi (16h - h^2) \Delta h \\
\text{work to lift slice } \approx 42\pi (16h - h^2)(21 - h) \Delta h \\
total \text{ work } = 42\pi \int_0^{11} (16h - h^2)(21 - h) \, dh
\]

\[
8^2 + (8 - h)^2 = r^2 + 64 - 16h + h^2 = 64 \\
r^2 = 16h - h^2
\]

11. Find the solution to \( \frac{dy}{dx} = \frac{\cos x}{y^2} \) that passes through (0,2). Use separation of variables.

\[
\int y^2 \, dy = \int \cos x \, dx \\
y^3 / 3 = \sin x + C \\
y^3 = 3 \sin x + D \\
y = \sqrt[3]{3 \sin x + D}
\]

When \( x = 0 \), we have \( y = 2 \), so \( 2 = \sqrt[3]{3 \sin 0 + D} \), or \( 2 = \sqrt[3]{D} \). Thus, \( D = 8 \).

Therefore, the solution is \( y = \sqrt[3]{3 \sin x + 8} \).