1. Consider the function \( f(x) = \frac{3}{5 - 2x} \).

(a) Is this function continuous on the interval \((\infty, \infty)\)? Explain.

No. \( f \) is discontinuous at \( x = 2.5 \), where \( f \) is undefined (and has a vertical asymptote).

(b) Compute the average rate of change of \( f \) on \([2, 2.01]\).

\[
\frac{f(2.01) - f(2)}{2.01 - 2} = \left[ \frac{3}{5 - 2(2.01)} - \frac{3}{5 - 2(2)} \right] \cdot \frac{1}{.01} \approx 6.122
\]

(c) Using the limit definition of the derivative, compute \( f'(x) \).

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{3}{5 - 2(x+h)} - \frac{3}{5 - 2x}}{h}
\]

\[
= \lim_{h \to 0} \frac{3(5-2x) - 3(5-2(x+h))}{h(5-2x)(5-2(x+h))}
\]

\[
= \lim_{h \to 0} \frac{15 - 6x - (15 - 6x - 6h)}{h(5-2(x+h)(5-2x))}
\]

\[
= \lim_{h \to 0} \frac{6h}{h(5-2(x+h)(5-2x))}
\]

\[
= \lim_{h \to 0} \frac{6}{5 - 2(x+h)(5-2x)}
\]

\[
= \frac{6}{(5 - 2x)^2}
\]

(d) Find the equation of the tangent line to \( f \) at \( x = 2 \).

We want \( y = mx + b \). \( m = f'(2) = \frac{6}{(5 - 2(2))^2} = 6 \), so \( y = 6x + b \).

[Note that this slope agrees well with our answer from (b) above.]

When \( x = 2 \), \( y = f(2) = \frac{3}{5 - 2(2)} = 3 \).

Thus, \( 3 = 6 \cdot 2 + b \), so \( b = -9 \) and we have \( y = 6x - 9 \).

2. Given that \( f(0) = 2 \), \( g(0) = 3 \), \( f'(0) = 5 \), \( g'(0) = 7 \), and \( f'(3) = \pi \) compute the following.

(a) \( h'(0) \) if \( h(z) = f(z)g(z) \)

\( h'(0) = f'(0)g(0) + f(0)g'(0) = (5)(3) + (2)(7) = 29 \)

(b) \( j'(0) \) if \( j(z) = \frac{f(z)}{g(z)} \)

\( j'(0) = \frac{f'(0)g(0) - f(0)g'(0)}{|g(0)|^2} = \frac{(5)(3) - (2)(7)}{3^2} = \frac{1}{9} \)

(c) \( k'(0) \) if \( k(z) = f(g(z)) \)

\( k'(0) = f'(g(0)) \cdot g'(0) = f'(3) \cdot (7) = (\pi)(7) = 7\pi \)
3. (a) Find $\frac{dy}{dt}$ if $y = t^5 + 5t + e^5 + \frac{t}{5} + \frac{5}{\sqrt{t}} + \ln (5t) + \arctan (5t) + \ln(5) + \sin 5$.

$$\frac{dy}{dt} = 5t^4 + (\ln 5)5t + 0 + \frac{1}{5} - 5t^{-2} + 5 \cdot \frac{1}{5} t^{-6/5} + \frac{1}{5t} \cdot 5 + \frac{1}{1 + (5t)^2} \cdot 5 + 0 + 0$$

$$= 5t^4 + (\ln 5)5t + \frac{1}{5} - \frac{5}{t^2} - \frac{1}{t^{5/5}} + \frac{1}{t} + \frac{5}{1 + 25t^2}$$

(b) Find $\frac{dy}{dx}$ if $y = \sqrt[3]{x} \cos(7x^3)$. 

$$\frac{dy}{dx} = \frac{1}{3}x^{-2/3} \cos(7x^3) + \sqrt[3]{x}(-\sin(7x^3)(21x^2)) = \frac{\cos(7x^3)}{3x^{2/3}} - 21x^{7/3} \sin(7x^3)$$

(c) Find $\frac{dy}{dz}$ if $y = \frac{e^z + e^{\pi}}{\tan 4 - 7z}$.

$$\frac{dy}{dz} = e^z(\tan 4 - 7z) - (-7)(e^z + e^{\pi}) \quad (\tan 4 - 7z)^2$$

(d) Find $\frac{dy}{dr}$ if $y = \tan(e^{\arcsin(5r)})$.

$$\frac{dy}{dr} = \sec^2(e^{\arcsin(5r)}) \cdot e^{\arcsin(5r)} \cdot \left(r^2 \cdot \frac{1}{\sqrt{1 - 25r^2}} \cdot 5 + 2r \arcsin(5r)\right)$$

(e) Find $\frac{dy}{dx}$ if $y^3 + yx^2 + x^2 = 3y^2$. Here we use implicit differentiation.

$$3y^2 \frac{dy}{dx} + \frac{dy}{dx} x^2 + 2xy + 2x = 6y \frac{dy}{dx}$$

$$3y^2 \frac{dy}{dx} + \frac{dy}{dx} x^2 - 6y \frac{dy}{dx} = -2xy - 2x$$

$$\frac{dy}{dx} (3y^2 + x^2 - 6y) = -2xy - 2x$$

$$\frac{dy}{dx} = \frac{-2xy - 2x}{3y^2 + x^2 - 6y}$$

(f) Find $\frac{dy}{dt}$ if $y = (1 + x^6)^{8x}$.

Since we have $x$ in the base and the exponent, we need logarithmic differentiation.

$$\ln y = 8x \ln(1 + x^6)$$

$$\frac{1}{y} \frac{dy}{dx} = 8 \cdot \ln(1 + x^6) + 8x \cdot \frac{1}{1 + x^6} \cdot 6x^5$$

$$\frac{dy}{dx} = \left[8 \cdot \ln(1 + x^6) + \frac{48x^6}{1 + x^6}\right] \cdot y$$

$$\frac{dy}{dx} = \left[8 \cdot \ln(1 + x^6) + \frac{48x^6}{1 + x^6}\right] \cdot (1 + x^6)^{8x}$$
4. Given the graph of \( f \), sketch a graph of \( f' \) and a graph of \( F \), an antiderivative of \( f \) such that \( F(0) = -1 \).

![Graphs of f, f', and F](image)

Note: The concave up portion on the left side of the graph of \( f \) is a perfect parabola, so its derivative \( (f') \) is linear; since you don’t know the equation for \( f \), your graph of \( f' \) may be concave up/down there.

5. Shown below is a graph of \( f' \) on its entire domain. The graph is NOT \( f \).

At which \( x \)-value(s) (if any)
(a) does \( f \) have a stationary point? \( k \)
(b) does \( f \) have a local max? \( k \)
(c) does \( f \) have a local min? none
(d) does \( f' \) have a stationary point? \( d, g, i, m \)
(e) does \( f' \) have a local max? \( d, i \)
(f) does \( f' \) have a local min? \( g, m \)

(b) \( f \) decreasing? \( k \) to \( n \)
(c) \( f' \) increasing? \( a \) to \( d \) and \( g \) to \( i \) and \( m \) to \( n \)
(d) \( f' \) decreasing? \( d \) to \( g \) and \( i \) to \( m \)
(e) \( f \) concave up? \( a \) to \( d \) and \( g \) to \( i \) and \( m \) to \( n \)
(f) \( f \) concave down? \( d \) to \( g \) and \( i \) to \( m \)

* Whether to include the endpoints of these intervals will depend on your instructor’s definitions.

![Graph of f'](image)

On what interval(s)* is
(a) \( f \) increasing? \( a \) to \( k \)
(b) \( f \) decreasing? \( k \) to \( n \)
(c) \( f' \) increasing? \( a \) to \( d \) and \( g \) to \( i \) and \( m \) to \( n \)
(d) \( f' \) decreasing? \( d \) to \( g \) and \( i \) to \( m \)
(e) \( f \) concave up? \( a \) to \( d \) and \( g \) to \( i \) and \( m \) to \( n \)
(f) \( f \) concave down? \( d \) to \( g \) and \( i \) to \( m \)

6. Solve the IVP \( y' = e^x - \sin x + 5 \) given that \( y(0) = 3 \).

We antidifferentiate each side to obtain \( y(x) = e^x + \cos x + 5x + C \). To find \( C \), we let \( x = 0 \), meaning \( 3 = e^0 + \cos 0 + 5 \cdot 0 + C \), so \( C = 1 \) and our solution is \( y(x) = e^x + \cos x + 5x + 1 \).
7. Evaluate the following limits.
Throughout this solution, the symbol ★ will stand for whatever notation your instructor prefers for using L’Hôpital’s Rule on the indeterminate form 0/0; this may be \( \frac{0}{0} \) or \( L' = \star \) or \( H = \star \) or \( 0/0 \) and so, by L’Hôpital’s Rule, is equal to” or something else. The symbol \( \triangledown \) will serve the same purpose for the indeterminate form \( \infty/\infty \).

(a) \( \lim_{x \to \infty} \frac{x^2}{\ln x} = \lim_{x \to \infty} \frac{2x}{1/x} = \lim_{x \to \infty} 2x^2 = \infty \)

(b) \( \lim_{z \to 0} \frac{\sin(5z) - 5z}{z^3} = \lim_{z \to 0} \frac{5 \cos(5z) - 5}{3z^2} \star \lim_{z \to 0} \frac{-25 \sin(5z)}{6z} \star \lim_{z \to 0} \frac{-125 \cos(12z)}{6} = -\frac{125}{6} \)

(c) \( \lim_{z \to 0} \frac{e^z - 1}{\cos z} = \frac{0}{1} = 0 \)

(d) \( \lim_{r \to 2} \frac{r^3 - 8}{r - 2} \star \lim_{r \to 2} \frac{3r^2}{1} = 12 \)

8. Consider the function \( f(x) = x^6 - 2x^3 \) on the interval \([-2, 2]\).

(a) Find the \( x \) and \( y \) coordinates of any and all critical points and use calculus to classify each as a local maximum, local minimum, or neither.

\[ f'(x) = 6x^5 - 6x^2 \]

Since \( f'(x) \) never fails to exist, we just need to solve \( f'(x) = 0 \).

\[ 0 = 6x^2(x^3 - 1) \]

\[ \Rightarrow x = 0, 1 \]

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<thead>
<tr>
<th>( x )</th>
<th>( f' )</th>
<th>( f )</th>
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<tbody>
<tr>
<td>( -2 \leq x &lt; 0 )</td>
<td>negative</td>
<td>positive</td>
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<tr>
<td>( 0 &lt; x &lt; 1 )</td>
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</tr>
<tr>
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<td>positive</td>
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\( y \)-values: \( f(0) = 0, f(1) = -1 \)

So, \( f \) has a local minimum at \((1, -1)\); \((0, 0)\) is not a local extremum.

(b) Find the \( x \) and \( y \) coordinates of any and all global extrema and classify each as a global maximum or global minimum.

We check the \( y \)-values at the local extrema and the endpoints.

\( y \)-values: \( f(-2) = 80, f(1) = -1, f(2) = 48 \)

So, \( f \) has a global minimum at \((1, -1)\) and a global maximum at \((-2, 80)\).

(c) Find the \( x \)-coordinate(s) of any and all inflection points.

\[ f''(x) = 30x^4 - 12x \]

Since \( f''(x) \) never fails to exist, we just need to solve \( f''(x) = 0 \).

\[ 0 = 6x(5x^3 - 2) \]

\[ \Rightarrow x = 0, \sqrt[3]{0.4} \]

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So, the \( x \)-values of the inflection points of \( f \) are \( x = 0 \) and \( x = \sqrt[3]{0.4} \).