1. Use a second-degree Taylor polynomial to estimate $\sqrt[3]{28}$.

We choose $f(x) = \sqrt[3]{x}$ and $x_0 = 27$ because 27 is the perfect cube closest to 28.

\[
f(x) = x^{1/3} \quad f(27) = 3
\]

\[
f'(x) = \frac{1}{3} x^{-2/3} \quad f'(27) = \frac{1}{3 \cdot 27^{2/3}} = \frac{1}{27}
\]

\[
f''(x) = \frac{2}{9} x^{-5/3} \quad f''(27) = \frac{2}{9 \cdot 27^{5/3}} = -\frac{2}{2187}
\]

Now plug in to the Taylor polynomial formula with $x_0 = 27$.

\[
P_2(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 = 3 + \frac{1}{27} (x - 27) - \frac{1}{2187} (x - 27)^2
\]

Finally, evaluate at $x = 28$.

\[
\sqrt[3]{28} \approx P_2(28) = 3 + \frac{1}{27} (28 - 27) - \frac{1}{2187} (28 - 27)^2 = \frac{6641}{2187} \approx 3.0365797
\]

2. What is the largest possible error that could have occurred in your previous estimate?

We know that $|f(x) - P_n(x)| \leq \frac{K_{n+1}}{(n+1)!}|x - x_0|^{n+1}$. 

In this case, $n = 2$, $x_0 = 27$, and $x = 28$.

\[
K_3 = \text{max of } |f'''(x)| \text{ on } [27, 28] = \text{max of } \left| \frac{10}{27x^{8/3}} \right| \text{ on } [27, 28] = \frac{10}{27 \cdot 27^{8/3}} = \frac{10}{177147}
\]

Putting this all together, we have $|f(x) - P_2(x)| \leq \frac{10}{177147} |28 - 27|^3 = \frac{5}{531441} \approx 0.0000094$.

3. Use a comparison to show whether each of the following converges or diverges. If an integral converges, give a good upper bound for its value.

(a) \[
\int_1^{\infty} \frac{7 + 5\sin x}{x^2} \, dx
\]

For all $x \geq 1$, we have $0 \leq \frac{7 + 5\sin x}{x^2} \leq \frac{7 + 5(1)}{x^2} = \frac{12}{x^2}$ because the maximum of $\sin x$ is 1.

\[
12 \int_1^{\infty} \frac{dx}{x^2} = 12 \lim_{t \to \infty} \int_1^{t} \frac{dx}{x^2} = 12 \lim_{t \to \infty} \left[ -\frac{1}{x} \right]_1^t = 12 \lim_{t \to \infty} \left[ -\frac{1}{t} - (-1) \right] = 12[0 - (-1)] = 12
\]

Therefore, the original integral in question must converge to a value less than 12.

(b) \[
\int_1^{\infty} \frac{1 + 3x^2 + 2x^3}{\sqrt{10x^{12} + 17x^{10}}} \, dx
\]

For $x \geq 1$, we have \[
\frac{1 + 3x^2 + 2x^3}{\sqrt{10x^{12} + 17x^{10}}} \geq \frac{2x^3}{\sqrt{10x^{12} + 17x^{10}}} \geq 0.
\]

(We've made the numerator smaller and the denominator larger, so the new fraction is smaller.)
4. Decide if each of the following sequences \( \{a_k\}_{k=1}^\infty \) converges or diverges. If a sequence converges, compute its limit.

(a) \( a_k = 3 + \frac{1}{10^k} \)

Terms are 3.1, 3.01, 3.001, 3.0001, ....

\[
\lim_{k \to \infty} \left( 3 + \frac{1}{10^k} \right) = 3,
\]

so the sequence converges to 3.

(b) \( a_k = (-1)^k \)

Terms are \(-1, 1, -1, 1, ....\)

\[
\lim_{k \to \infty} (-1)^k
\]

doesn’t exist, so the sequence diverges.

(c) \( a_k = \frac{5e^k}{7e^k + \ln(k+1)} \)

By L’Hopital’s Rule,

\[
\lim_{k \to \infty} \frac{5e^k}{7e^k + \ln(k+1)} = \lim_{k \to \infty} \frac{5e^k}{7e^k + \frac{1}{k+1}} = \frac{5}{7},
\]

so the sequence converges to \( \frac{5}{7} \).

5. Decide if each of the following series converges or diverges. If a series converges, find its value.

(a) \( 3.1 + 3.01 + 3.001 + 3.0001 + ... \)

\[
\lim_{k \to \infty} a_k = 3 \neq 0,
\]

so the series diverges by the nth Term Test. (We keep adding 3’s forever.)

[Compare this with the first sequence of the previous problem.]

(b) \( 1 + 1/2 + 1/3 + 1/4 + ... \)

This is the famous Harmonic Series, which diverges even though the terms do approach 0. We can use the Integral Test:

\[
\int_1^\infty \frac{1}{x} \, dx
\]

diverges, which means that \( \sum_{k=1}^\infty \frac{1}{k} \) must diverge too.

(c) \( 5 - 5/3 + 5/9 - 5/27 + ... \)

This is a geometric series with \( r = \frac{1}{3} \), so it converges to

\[
\frac{a}{1-r} = \frac{5}{1-(-1/3)} = \frac{15}{4}.
\]

6. Decide if each of the following series converges or diverges. If a series converges, find upper and lower bounds for its value.

(a) \( \sum_{k=1}^\infty \frac{(-1)^k}{\sqrt{k+1}} \)

[Alternating Series Test]

The terms of this series alternate in sign.

And, \( \frac{1}{\sqrt{2}} \geq \frac{1}{\sqrt{3}} \geq \frac{1}{\sqrt{4}} \geq ... \geq 0 \). (Or, more formally, \( k < k+1 \Rightarrow (k+1) < (k+1) + 1 \Rightarrow \sqrt{(k+1)} < \sqrt{(k+1) + 1} \Rightarrow \frac{1}{\sqrt{(k+1)}} > \frac{1}{\sqrt{(k+1) + 1}} \) so the terms are decreasing in size.)

And, \( \lim_{k \to \infty} \frac{1}{\sqrt{k+1}} = 0 \).

Therefore, by the Alternating Series Test, the series must converge.

We know that any two consecutive partial sums will provide upper and lower bounds:

\[
\text{lower bound } = S_1 = -\frac{1}{\sqrt{2}}, \quad \text{upper bound } = S_2 = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}}
\]

[To get better bounds, use later partial sums, such as \( S_5 \) and \( S_6 \).]
(b) \[ \sum_{k=1}^{\infty} \frac{(2k)!}{3^k(k!)^2} \]  

\[
\lim_{k \to \infty} \frac{a_{k+1}}{a_k} = \frac{\frac{(2(k+1))!}{3^{k+1}((k+1)!)^2}}{\frac{(2k)!}{3^k(k!)^2}} \\
= \lim_{k \to \infty} \frac{(2k + 2)!}{(2k)!} \frac{3^k}{3^{k+1}} \frac{(k!)^2}{((k+1)!)^2} \\
= \lim_{k \to \infty} \left( \frac{2k + 2}{2(k + 1)} \right) \frac{1}{1} \frac{1}{3} \frac{1}{(k + 1)^2} \\
= \frac{4k^2 + 6k + 2}{3k^2 + 6k + 3} \\
= \frac{4}{3}
\]

Since the limit of the ratio is greater than 1, the series diverges.

(c) \[ \sum_{k=1}^{\infty} \left( \frac{1}{100} + \frac{1}{k^5} \right) \]  

\[
\lim_{k \to \infty} \left( \frac{1}{100} + \frac{1}{k^5} \right) = \frac{1}{100} \neq 0, \text{ so, by the nth Term Test, the series diverges.}
\]

(d) \[ \sum_{k=1}^{\infty} \frac{\sqrt{9k^8 + 5k^6}}{12k^5 + 3} \]  

[Comparison Test]

For \( k \geq 1 \), we have \( \frac{\sqrt{9k^8 + 5k^6}}{12k^5 + 3} > \frac{\sqrt{9k^8}}{12k^5 + 3k^5} > 0 \).

But, \( \frac{\sqrt{9k^8}}{12k^5 + 3k^5} = \frac{3k^4}{15k^5} = \frac{1}{5} \frac{1}{k} \) and we know that \( \sum_{k=1}^{\infty} \frac{1}{k} \) diverges (use Integral Test or note that \( p = 1 \)).

Therefore, the original series, which has larger terms, must diverge also.

(e) \[ \sum_{k=2}^{\infty} \frac{1}{k(\ln(k))^2} \]  

[Integral Test]

\[
\int_2^{\infty} \frac{1}{x(\ln x)^2} \, dx = \lim_{t \to \infty} \int_2^{t} \frac{1}{x(\ln x)^2} \, dx \\
= \lim_{t \to \infty} \int_{x=2}^{x=t} \frac{1}{u^2} \, du \quad \text{Substitute } u = \ln x, \text{ so } du = \frac{dx}{x}, \\
= \lim_{t \to \infty} u^{-1} \bigg|_{x=2}^{x=t} \\
= \lim_{t \to \infty} \left[ -1 \ln t - \frac{1}{\ln 2} \right] \\
= 0 - \frac{1}{\ln 2} \\
= \frac{1}{\ln 2}
\]

The integral converges, so the series must converge too.
Further, we know that \( \int_2^\infty \frac{1}{x(\ln x)^2} \, dx \leq \sum_{k=2}^\infty \frac{1}{k(\ln k)^2} \leq a_2 + \int_2^\infty \frac{1}{x(\ln x)^2} \, dx \).

There \( \text{therefore, our lower bound is} \) \( \int_2^\infty \frac{1}{x(\ln x)^2} \, dx = \frac{1}{\ln 2} \).

And our upper bound is \( \sum_{k=2}^\infty \frac{1}{k(\ln k)^2} = \sum_{k=2}^\infty \frac{1}{k(\ln k)^2} + \frac{1}{\ln 2} \).

7. Does the first series from the previous problem converge absolutely or conditionally?

\[ \sum_{k=1}^\infty \frac{(-1)^k}{\sqrt{k + 1}} = \sum_{k=1}^\infty \frac{1}{\sqrt{k + 1}}, \text{ which diverges by the Integral Test (check for yourself).} \]

Therefore, the first series from the previous problem converges conditionally.

8. Compute the radius and interval (including endpoints) of convergence for \( \sum_{k=1}^\infty \frac{(x + 3)^k}{k \cdot 5^k} \).

\[
\lim_{k \to \infty} \left| \frac{(x + 3)^{k+1}}{(x + 3)^k \cdot k \cdot 5^k} \right| = \lim_{k \to \infty} \left| \frac{(x + 3)^{k+1}}{k + 1} \cdot \frac{5^k}{5^{k+1}} \right|
\]

Use L'Hopital on the middle fraction.

\[
= \left| (x + 3) \cdot \frac{1}{5} \right| \quad \text{So, we are guaranteed convergence when} \quad \left| \frac{x + 3}{5} \right| < 1. \quad \text{But this is equivalent to the following.}
\]

\[
-1 < \frac{x + 3}{5} < 1
\]

\[
-5 < x + 3 < 5
\]

\[
-8 < x < 2
\]

To check convergence at the endpoints (where the Ratio Test is inconclusive), we plug in to the series itself.

At \( x = 2 \), we have \( \sum_{k=1}^\infty \frac{(2 + 3)^k}{k \cdot 5^k} = \sum_{k=1}^\infty \frac{5^k}{k \cdot 5^k} = \sum_{k=1}^\infty \frac{1}{k} \), which is the Harmonic Series and thus diverges.

At \( x = -8 \), we have \( \sum_{k=1}^\infty \frac{(-8 + 3)^k}{k \cdot 5^k} = \sum_{k=1}^\infty \frac{(-5)^k}{k \cdot 5^k} = \sum_{k=1}^\infty \frac{(-1)^k}{k} \), which converges by the Alternating Series Test.

Thus, the interval of convergence is \( -8 \leq x < 2 \) and the radius of convergence is 5.

9. Evaluate the following exactly.

(a) \( 1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} + \frac{1}{120} + \cdots \)

We recall that \( e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = 1 + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots \).

The series in question is the series for \( e^x \) with \( x = -1 \); therefore, it converges to \( e^{-1} \).

(b) \( \frac{8}{3} - \frac{8}{9} + \frac{8}{27} - \frac{8}{81} + \cdots \)

We recognize this as a geometric series with \( r = -1/3 \) and \( a = 8/3 \); therefore, it converges to

\[
\frac{a}{1 - r} = \frac{8/3}{1 - (-1/3)} = 2.
\]
\(1 - \frac{\pi^2}{2} + \frac{\pi^4}{24} - \frac{\pi^6}{720} + \ldots\)

We recall that \(\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \ldots\)

The series in question is the series for \(\cos x\) with \(x = \pi\); therefore, it converges to \(\cos \pi\), which is \(-1\).

10. (a) Using summation notation, write the series equal to \(\int_0^1 e^{-x^2} \, dx\) and show that it converges.

We know \(e^w = 1 + \frac{w^2}{2!} + \frac{w^3}{3!} + \ldots\), so by substitution we obtain the following.

\[e^{-x^2} = 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \ldots = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \ldots\]

Thus, \(\int_0^1 e^{-x^2} \, dx = \int_0^1 \left[1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \ldots\right] dx\)

\[= \left[x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \ldots\right]_0^1\]

\[= 1 - \frac{1^3}{3} + \frac{1^5}{5 \cdot 2!} - \frac{1^7}{7 \cdot 3!} + \ldots\]

\[= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k + 1)k!}\]

The terms of this series alternate in sign.

And, \(1 \geq \frac{1^3}{3} \geq \frac{1^5}{5 \cdot 2!} \geq \frac{1}{7 \cdot 3!} \geq \ldots \geq 0\).

And, \(\lim_{k \to \infty} \frac{1}{(2k + 1)k!} = 0\).

Therefore, by the Alternating Series Test, the series must converge.

(b) If \(f(x) = e^{-x^2}\), what is \(f^{(400)}(0)\)? What is \(f^{(401)}(0)\)?

In the previous part, we found the following.

\[e^{-x^2} = 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \ldots = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \ldots\]

We know that \(f^{(400)}(0)\) will appear in the Taylor series for \(f(x)\) in the coefficient of the \(x^{400}\) term, which is \(\frac{f^{(400)}(0)}{400!} x^{400}\).

From above, we see that term is \(\frac{x^{400}}{200!}\).

Setting them equal gives \(\frac{f^{(400)}(0)}{400!} x^{400} = \frac{x^{400}}{200!}\), which means that \(f^{(400)}(0) = \frac{400!}{200!}\).

[Note that this does not simplify to 200! Rather, \(\frac{400!}{200!} = 400 \cdot 399 \ldots \cdot 202 \cdot 201\).

We know that \(f^{(401)}(0)\) will appear in the Taylor series for \(f(x)\) in the coefficient of the \(x^{401}\) term, which is \(\frac{f^{(401)}(0)}{401!} x^{401}\).

However, the Taylor series above for \(f(x)\) has no terms with odd powers of \(x\), meaning that the coefficients of all those terms must be 0.

Therefore, we know that \(f^{(401)}(0) = 0\).