Mathematics 105
Sections A, B, C, and D
Final Exam
Dec 13, 2011

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- You must show all work to receive credit.
- No electronic devices other than calculators are permitted.
- Give exact answers (such as \ln 5 or \(e^2\)) unless requested otherwise.
- Partial credit will be assigned based upon the correctness, completeness, and clarity of your answers.
- Correct answers without proper justification or those that use unapproved short-cut methods will not receive full credit.
- If you use a calculator to help find an answer, you must write down enough information on what you have done to make your method understandable.
1. Consider the function:

\[ g(x) = \begin{cases} 
  x^3 - 1 & \text{if } x \geq 0 \\
  1 - x^3 & \text{if } x < 0
\end{cases} \]

(a) What is \( g'(x) \)?

\[ g'(x) = \begin{cases} 
  3x^2 & \text{if } x > 0 \\
  -3x^2 & \text{if } x < 0
\end{cases} \]

* See note in (g) below for why \( g'(0) \) is undefined

(b) Evaluate \( \lim_{x \to 0^+} g(x) \).

\[ \lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} x^3 - 1 = -1 \]

(c) Evaluate \( \lim_{x \to 0^-} g(x) \).

\[ \lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} 1 - x^3 = 1 \]

(d) Is \( g(x) \) continuous at \( x = 0 \)?

No, since \( \lim_{x \to 0^+} g(x) \neq \lim_{x \to 0^-} g(x) \)

Thus, \( \lim_{x \to 0} g(x) \) DNE \( \Rightarrow g \) is not continuous at \( x = 0 \)

(e) Evaluate \( \lim_{x \to 0^+} g'(x) \).

\[ \lim_{x \to 0^+} g'(x) = \lim_{x \to 0^+} 3x^2 = 0 \]

(f) Evaluate \( \lim_{x \to 0^-} g'(x) \).

\[ \lim_{x \to 0^-} g'(x) = \lim_{x \to 0^-} -3x^2 = 0 \]

(g) In light of the above, check your answer to part (a). Does \( g'(0) \) exist? Why or why not?

Despite fact that \( \lim_{x \to 0^+} g'(x) = \lim_{x \to 0^-} g'(x) = 0 \), \( g'(0) \) does not exist since \( g \) is not continuous at \( x = 0 \) so \( g'(0) \) DNE

\[ g \text{ has a hole at } x = 0 \]
2. Evaluate \( \int_{-1}^{1} \left( t^2 + \frac{1}{1+t^2} \right) dt \).

\[
\int_{-1}^{1} \left( t^2 + \frac{1}{1+t^2} \right) dt = \left[ \frac{t^3}{3} + \arctan t \right]_{-1}^{1}
= \left( \frac{1}{3} + \arctan 1 \right) - \left( -\frac{1}{3} + \arctan (-1) \right)
= \frac{2}{3} + \frac{\pi}{4} + \frac{\pi}{4} = \frac{4 + 3\pi}{6}
\]

3. Find the solution to the initial value problem where \( y' = x^2 + 2x \ln 2 \) with \( y(0) = 2 \).

**Antidifferentiate to find general solution:**

\[
y = \frac{x^3}{3} + 2^x + C
\]

**Use initial value \( y(0) = 2 \) to find \( C \):**

\[
2 = \frac{0^3}{3} + 2^0 + C \quad \Rightarrow \quad 2 = 1 + C \quad \Rightarrow \quad C = 1
\]

**Solution to IVP:**

\[
y = \frac{x^3}{3} + 2^x + 1
\]
4. Consider

\[ f(x) = \begin{cases} \quad a + bx^2 & \text{if } x < 2 \\ -x^2 + 10x - 4 & \text{if } x \geq 2 \end{cases} \]

(a) What condition(s) must be placed on the constants \(a\) and \(b\) in order for \(f\) to be continuous on \((-\infty, \infty)\)?

Since \(a+bx^2\) and \(-x^2+10x-4\) are polynomials, they are continuous for all real values of \(x\). Thus, the only possible point of discontinuity for \(f(x)\) is at \(x=2\), where the piecewise function changes definition on its domain.

So we need \(\lim_{x \to 2^-} f(x) = f(2)\)

\[
\begin{align*}
\lim_{x \to 2^-} f(x) &= \lim_{x \to 2^-} a + bx^2 = a + 4b \\
\lim_{x \to 2^+} f(x) &= \lim_{x \to 2^+} -x^2 + 10x - 4 = -4 + 20 - 4 = 12
\end{align*}
\]

for continuity we require \(a + 4b = 12\)

\[ f(x) = -2x + 10 \]

\(\therefore f\) is continuous if \(a + 4b = 12\)

(b) For what values of the constants \(a\) and \(b\) will \(f\) be differentiable on \((-\infty, \infty)\)?

\[
\begin{cases}
2bx & \text{if } x < 2 \\
-2x + 10 & \text{if } x \geq 2
\end{cases}
\]

\[
\begin{align*}
\lim_{x \to 2^-} f'(x) &= \lim_{x \to 2^-} 2bx = 4b \\
\lim_{x \to 2^+} f'(x) &= \lim_{x \to 2^+} -2x + 10 = 6
\end{align*}
\]

Need left \& right-hand limits to be equal in order for \(\lim_{x \to 2} f'(x)\) to exist

\[ \text{we require } 4b = 6 \implies b = \frac{3}{2} \]

\(f\) continuous at \(x = 2\) requires \(a + 4b = 12\)

\(\implies a + 4 \left(\frac{3}{2}\right) = 12\)

\(\implies a = 6\)

\[ f = \begin{cases} 6 & \text{if } x < 2 \\ -2x + 10 & \text{if } x \geq 2 \end{cases} \]

Note \(a = 6, b = \frac{3}{2}\) then \(f(x) = \begin{cases} 6 + \frac{3}{2}x^2 & \text{if } x < 2 \\ -x^2 + 10x - 4 & \text{if } x \geq 2 \end{cases} \)

\[ f'(x) = \begin{cases} 3x & \text{if } x < 2 \\ -2x + 10 & \text{if } x \geq 2 \end{cases} \]

Where \(f\) is continuous on \((-\infty, -2)\) and \(f\) is differentiable on \((-\infty, 0)\)

\(\therefore \lim_{x \to 2^-} f(x) = 12 = f(2)\)

\(\therefore \lim_{x \to 2^+} f'(x) = 6 = f'(2)\)
5. A clock on the wall reads 10:00. The hour hand is \( h = 5 \) ft long and the minute hand is \( m = 7 \) ft long. The distance between to two tips is \( z \). The angle between the two hands is \( \theta \).

(a) The law of cosines states \( h^2 + m^2 - 2hm \cos \theta = z^2 \). Find \( z \) for this problem. [Hint: start by finding \( \theta \) when the clock reads 10:00.]

\[
\begin{align*}
\text{Note: at 10:00, } \theta &= 2 \left( \frac{\pi}{12} \right) = \frac{\pi}{6} \\
5^2 + 7^2 - 2(5)(7)\cos \left( \frac{\pi}{6} \right) &= z^2 \\
25 + 49 - 35 &= z^2 \\
39 &= z^2 \\
z &= \sqrt{39} \approx 6.24 \text{ ft}
\end{align*}
\]

(b) Explain why \( \frac{d\theta}{dt} = \frac{11\pi}{6} \) radians per hour.

- \( h \) moves \( 2\pi \) radians per hour
- \( m \) moves \( \frac{2\pi}{12} \) radians per hour

\[
\frac{d\theta}{dt} = 2\pi - \frac{\pi}{12} = \frac{23\pi}{12} = \frac{11\pi}{6} \text{ radians per hour}
\]

(c) Find \( \frac{dz}{dt} \).

Implicity differentiate \( 25 + 49 - 70 \cos \theta = z^2 \) with respect to \( t \)

\[
\begin{align*}
+70 \sin \theta \frac{d\theta}{dt} &= 2z \frac{dz}{dt} \\
70 \sin \left( \frac{\pi}{6} \right) \cdot \frac{4\pi}{6} &= 2 \sqrt{39} \frac{dz}{dt} \\
\frac{dz}{dt} &= \frac{70 \sqrt{39} \pi^2}{12} \cdot \frac{1}{2.59} \approx 27.95 \text{ ft/hr}
\end{align*}
\]
6. Consider \( f(x) = x^3 - x \), Newton’s method generates successive estimates in finding a root of the equation \( f(x) = 0 \) using the formula \( x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \) based upon an initial guess \( x_0 \).

(a) Starting with an initial guess of \( x_0 = \frac{1}{\sqrt{5}} \), use Newton’s method to generate the first four estimates, \( x_1, x_2, x_3, \) and \( x_4 \), to a root of \( x^3 - x = 0 \).

\[
\frac{f'}{f}(x) = 3x^2 - 1
\]

\[
x_{k+1} = x_k - \frac{x_k^3 - x_k}{3x_k^2 - 1} = \frac{2x_k^3}{3x_k^2 - 1}
\]

\[
x_0 = \frac{1}{\sqrt{5}}
\]

\[
x_1 = \frac{2(3x_0^2)}{3(3x_0^2 - 1)} = \frac{2}{3 \sqrt{5}} - \frac{\sqrt{5}}{2} = \frac{1}{\sqrt{5}}
\]

\[
x_2 = \frac{2(3x_1^2)}{3(3x_1^2 - 1)} = \frac{2}{3 \sqrt{5}} - \frac{\sqrt{5}}{2} = \frac{1}{\sqrt{5}}
\]

\[
x_3 = \frac{2(3x_2^2)}{3(3x_2^2 - 1)} = \frac{2}{3 \sqrt{5}} - \frac{\sqrt{5}}{2} = \frac{1}{\sqrt{5}}
\]

\[
x_4 = \frac{2(3x_3^2)}{3(3x_3^2 - 1)} = \frac{2}{3 \sqrt{5}} - \frac{\sqrt{5}}{2} = \frac{1}{\sqrt{5}}
\]

(b) Annotate the graph below to help demonstrate the geometric idea behind Newton’s method for finding the estimates in part (a). You do not need to give a complete derivation of the formula, but you must informally describe how the method generates estimates.

---

The x-intercept for line tangent to \( f \) at \( (x_k, f(x_k)) \) is \( x_k \).

The x-intercept for line tangent to \( f \) at \( (x_k, f(x_k)) \) is \( x_k \), and so on...

Thus creates the cycle: \( x_0 = \frac{1}{\sqrt{5}} \rightarrow x_1 = \frac{1}{\sqrt{5}} \rightarrow x_2 = \frac{1}{\sqrt{5}} \rightarrow x_3 = \frac{1}{\sqrt{5}} \rightarrow x_4 = \frac{1}{\sqrt{5}} \rightarrow ... \)
7. Consider two functions,

\[ f(x) = \int_{-3}^{x} \sin \sqrt{t} dt \quad \text{and} \quad g(x) = \int_{3}^{x} \sin \sqrt{t} dt \]

(a) What is the derivative of \( g(x) \)?

Since \(-3 \) is not in the domain of \( \sin \sqrt{t} \), it follows from the Fundamental Theorem that

\[ \frac{d}{dx} \int_{-3}^{x} \sin \sqrt{t} dt = \sin \sqrt{x} \]

(b) What is the derivative of \( f(x) \)?

\[ \frac{d}{dx} \int_{3}^{x} \sin \sqrt{t} dt \] does not exist since \(-3\) is not in the domain of \( \sin \sqrt{t} \)

(c) Why do your two answers differ?

Since \( \sin \sqrt{x} \) does not exist for \( x < 0 \)

8. Suppose \( f(0) = 4, \ f'(0) = 3, \ g(0) = -7, \ g(4) = 1, \) and \( g'(0) = \frac{\pi}{2} \). Compute the following or explain why an answer does not exist.

(a) \( h'(0) \) if \( h(x) = f(x)g(x) \)

\[ h'(x) = f'(x)g(x) + f(x)g'(x) \]

\[ \Rightarrow h'(0) = f'(0)g(0) + f(0)g'(0) = 3(-7) + 4(\frac{\pi}{2}) = -21 + 2\pi \]

(b) \( k'(0) \) if \( k(x) = \frac{f(x)}{g(x)} \)

\[ k'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2} \]

\[ k'(0) = \frac{f'(0)g(0) - f(0)g'(0)}{g(0)^2} = \frac{2(-7) - 4(\frac{\pi}{2})}{(-7)^2} = -21 - 2\pi \]

\[ \frac{2}{49} \]

(c) \( v'(0) \) if \( v(x) = e^{g(f(x))} \)

\[ v(x) = e^{g(f(x))} \cdot g'(f(x)) \cdot f'(x) \]

\[ v'(0) = e^{g(f(0))} \cdot g'(f(0)) \cdot f'(0) = e^{g(4)} \cdot g'(4) \cdot 3 = 3e \cdot g'(4) \]
9. A rectangular plot of farmland will be bounded on one side by a river and on the other three sides by a single-strand electric fence. With 800 meters of wire at your disposal, what is the largest area you can enclose?

Maximize Area: \( A = xy \)

Constraint Equation: \( P = 800 = 2x + y \) \( \Rightarrow y = 800 - 2x \)

\( A = xy = x(800 - 2x) = 800x - 2x^2 \)

\( A'(x) = 800 - 4x \), so \( A'(x) = 0 \) when \( 800 - 4x = 0 \) \( x = 200 \)

2nd Derivative Test: \( A''(x) = -4 < 0 \) \( \Rightarrow A''(200) = -4 < 0 \) \( \Rightarrow A \) has max at \( x = 200 \)

If \( x = 200 \), \( y = 800 - 2(200) = 400 \)

Therefore, maximum area is \( A = xy = 200 \cdot 400 = 80,000 \text{ ft}^2 \)

10. Consider the following limit

\[ \lim_{x \to \infty} \frac{\sqrt{x + 4}}{\sqrt{4x + 1}} \]

(a) What happens if you apply L'Hôpital's Rule? Be sure to apply the rule more than once.

\[ \lim_{x \to \infty} \frac{\sqrt{x + 4}}{\sqrt{4x + 1}} = \lim_{x \to \infty} \frac{\frac{1}{2\sqrt{x + 4}}}{\frac{4}{2\sqrt{4x + 1}}} = \lim_{x \to \infty} \frac{\frac{4}{2\sqrt{4x + 1}}}{\frac{2\sqrt{4x + 1}}{2\sqrt{4x + 1}}} = \lim_{x \to \infty} \frac{4}{2\sqrt{4x + 1}} \]

We're back to where we started, so L'Hôpital's rule is useless.

(b) Evaluate this limit algebraically.

\[ \lim_{x \to \infty} \frac{\sqrt{x + 4}}{\sqrt{4x + 1}} = \lim_{x \to \infty} \frac{\sqrt{x + 4}}{\sqrt{4x + 1}} = \left( \lim_{x \to \infty} \frac{x + 4}{4x + 1} \right)^{\frac{1}{2}} = \left( \lim_{x \to \infty} \frac{4x + 1}{4x + 1} \right)^{\frac{1}{2}} = \left( \frac{1}{4} \right)^{\frac{1}{2}} = \frac{1}{2} \]
11. This problem will evaluate an integral using the limit definition. We will evaluate 
\[ \int_{0}^{1} x^3 \, dx. \]

(a) If we partition the interval \([0, 1]\) into \(n\) equal length subintervals, how long is each interval? This is \(\Delta x_i\) for all \(i\).

\[ \Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n} \]

(b) Write a formula for the right end-point of the \(i^{th}\) interval. This is \(c_i\) for all \(i\).

\[ c_i = 0 + i(\Delta x) = 0 + i\left(\frac{1}{n}\right) = \frac{i}{n} \]

(c) Write the integral as the limit of a sum.

\[ R_n = \sum_{i=1}^{n} f(c_i) \Delta x = \sum_{i=1}^{n} \left(\frac{i}{n}\right)^3 \frac{1}{n} = \frac{1}{n^4} \sum_{i=1}^{n} i^3 \]

\[ \int_{0}^{1} x^3 \, dx = \lim_{n \to \infty} R_n = \lim_{n \to \infty} \frac{1}{n^4} \sum_{i=1}^{n} i^3 \]

(d) Using the fact that \(\sum_{i=1}^{n} i^3 = \frac{n^4 + 2n^3 + n^2}{4}\) evaluate the limit.

\[ \int_{0}^{1} x^3 \, dx = \lim_{n \to \infty} \frac{1}{n^4} \sum_{i=1}^{n} i^3 \]

\[ = \lim_{n \to \infty} \frac{1}{n^4} \left(\frac{n^4 + 2n^3 + n^2}{4}\right) \]

\[ = \lim_{n \to \infty} \frac{1}{4} \left(1 + \frac{2}{n} + \frac{1}{n^2}\right) = \frac{1}{4} \]

(e) Is your answer consistent with the answer the Fundamental Theorem of Calculus gives?

Yes, since

\[ \int_{0}^{1} x^3 \, dx = \frac{x^4}{4} \bigg|_{0}^{1} = \frac{1}{4} - 0 = \frac{1}{4} \]
12. Consider a continuous function $f$ defined on the interval $(0, \infty)$ with the following characteristics.

- $f(3) = 0$, $f'(2) = 0$, $f'(5) = 0$, $f'(3)$ is undefined, $f''(2) = 0$, and $f''(6) = 0$
- $f'(x) < 0$ on $(0, 2) \cup (2, 3) \cup (5, \infty)$, but $f'(x) > 0$ on $(3, 5)$
- $f''(x) > 0$ on $(0, 2) \cup (6, \infty)$, but $f''(x) < 0$ on $(2, 3) \cup (3, 6)$
- $\lim_{x \to \infty} f(x) = -3$ and $\lim_{x \to 0^+} f(x) = \infty$

It may help to do part (e) first.

(a) Identify all critical points for $f$. Classify each as a local maximum, local minimum, or neither.

(b) Identify all inflection points for the graph of $f$, if they exist. Justify your answer.

(c) Does $f$ have a global maximum on $(0, \infty)$? If so, what is it?

(d) Does $f$ have a global minimum on $(0, \infty)$? If so, what is it?

(e) Sketch a graph of $f$ below.