1. Find the following. [See Review for Exam II for integration tips and strategies.]

(a) Let \( u = x^3 \), so \( du = 3x^2 \, dx \) and \( du/3 = x^2 \, dx \).

\[
\int 12x^2 \cos(x^3) \, dx = 12 \int \cos(x^3) \, x^2 \, dx
\]
\[
= 12 \int \cos(u) \frac{du}{3}
\]
\[
= 4 \sin(u) + C
\]
\[
= 4 \sin(x^3) + C
\]

(b) We’ll use integration by parts: \( u = x \Rightarrow du = dx \) and \( dv = e^{-3x} \Rightarrow v = \frac{e^{-3x}}{-3} \).

\[
\int_0^\infty xe^{-3x} \, dx = \lim_{t \to \infty} \int_0^t xe^{-3x} \, dx
\]
\[
= \lim_{t \to \infty} \left[ \frac{e^{-3x} \cdot t}{-3} \bigg|_0^t - \int_0^t \frac{e^{-3x}}{-3} \, dx \right]
\]
\[
= \lim_{t \to \infty} \left[ \frac{-x}{3e^{3x}} - \frac{1}{9e^{3x}} \right]_0^t
\]
\[
= \lim_{t \to \infty} \left[ -\frac{-t}{3e^{3t}} - \frac{1}{9e^{3t}} \right] - \left[ \frac{0}{3e^0} - \frac{1}{9e^0} \right]
\]
\[
= (0 - 0) - (0 - 1/9)
\]
\[
= 1/9
\]

So, the integral converges (to this value).

(c) This integral is improper at \( x = 4 \) because the integrand has a vertical asymptote there, so we split into two integrals.

\[
\int_0^6 \frac{dx}{(x-4)^2} = \int_0^4 \frac{dx}{(x-4)^2} + \int_4^6 \frac{dx}{(x-4)^2}
\]
\[
= \lim_{a \to 4^-} \int_0^a \frac{dx}{(x-4)^2} + \lim_{b \to 4^+} \int_b^6 \frac{dx}{(x-4)^2}
\]
\[
= \lim_{a \to 4^-} \left[ -\frac{1}{(x-4)} \right]|_0^a + \lim_{b \to 4^+} \left[ -\frac{1}{(x-4)} \right]|_b^6 \int u^{-2} \, du = -u^{-1} + C
\]
\[
= \lim_{a \to 4^-} \left[ -\frac{1}{(a-4)} - \frac{-1}{0-4} \right] + \lim_{b \to 4^+} \left[ -\frac{1}{(b-4)} - \frac{-1}{6-4} \right]
\]
\[
= \infty \quad \text{and} \quad \infty
\]

Since \( \lim_{a \to 4^-} \frac{-1}{(a-4)} = \infty \) and \( \lim_{b \to 4^+} \frac{-1}{(b-4)} = \infty \), this integral diverges (to \( \infty \)).

(d) Partial Fractions:

Write \( \frac{3x^2 + 2x - 5}{(x^2+1)(x-4)} = \frac{Ax + B}{x^2+1} + \frac{C}{x-4} \). Now multiply both sides by \((x^2+1)(x-4)\) to get

\[3x^2 + 2x - 5 = (Ax + B)(x-4) + C(x^2 + 1)\]

Let \( x = 4 \). Then \( 51 = C(17) \), so \( C = 3 \).

Let \( x = 0 \). Then \( -5 = B(-4) + 3(1) \), so \( B = 2 \).
Let \( x = 1 \). Then 0 = \((A(1) + 2)(-3) + 3(2)\), so \( A = 0 \).

\[
\int \frac{3x^2 + 2x - 5}{(x^2 + 1)(x - 4)} \, dx = \int \left[ \frac{2}{x^2 + 1} + \frac{3}{x - 4} \right] \, dx
= 2 \arctan x + 3 \ln |x - 4| + D
\]

(e) Let \( u = \sec x \), so \( du = \sec x \tan x \, dx \).

New limits: \( x = 0 \Rightarrow u = \sec 0 = 1 \) and \( x = \pi/3 \Rightarrow u = \sec(\pi/3) = 2 \).

\[
\int_0^{\pi/3} \tan^3 x \sec^6 x \, dx = \int_0^{\pi/3} \tan^2 x \sec^4 x \sec x \tan x \, dx
\]

Break off a \( \sec x \tan x \).

\[
= \int_0^{\pi/3} (\sec^2 x - 1) \sec^4 x \sec x \tan x \, dx \quad \text{Use} \quad \tan^2 x = \sec^2 x - 1.
\]

\[
= \int_1^2 (u^2 - 1)u^4 \, du \quad \text{Change the limits. See above.}
\]

\[
= \int_1^2 (u^6 - u^4) \, du
\]

\[
= \left[ \frac{u^7}{7} - \frac{u^5}{5} \right]_1
\]

\[
= \frac{2^7}{7} - \frac{2^5}{5} - \left[ \frac{1^7}{7} - \frac{1^5}{5} \right]
\]

\[
= 418 \frac{35}{35}
\]

This is about 11.943.

(f) Let \( x = 5 \sin t \), so \( dx = 5 \cos t \, dt \).

\[
2^2 + y^2 = 5^2 \Rightarrow y = \sqrt{25 - x^2}
\]

\[
\sin t = \frac{\text{opp}}{\text{hyp}} = \frac{x}{5} \Rightarrow t = \arcsin(x/5)
\]

\[
\cos t = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{25 - x^2}}{5} \Rightarrow 5 \cos t = \sqrt{25 - x^2}
\]

\[
\int \sqrt{25 - x^2} \, dx = \int 5 \cos t \cdot 5 \cos t \, dt = \int 25 \cos^2 t \, dt
\]

Use \( dx \) and \( \cos t \) from above.

\[
= \int 25 \cos^2 t \, dt
\]

\[
= 25 \int \left[ \frac{1 + \cos(2t)}{2} \right] \, dt \quad \text{Use} \quad \cos^2 t = \frac{1 + \cos(2t)}{2} \quad \text{or} \quad \text{table #42}.
\]

\[
= 25 \left[ \frac{t}{2} + \frac{\sin(2t)}{4} \right] + C
\]

\[
= 25 \left[ \frac{\arcsin(x/5)}{2} + \frac{2 \sin t \cos t}{4} \right] + C \quad \text{Use} \quad \sin(2t) = \sin t \cos t \quad \text{and} \quad x \quad \text{from above.}
\]

\[
= 25 \left[ \frac{\arcsin(x/5)}{2} + \frac{x \cdot \sqrt{25 - x^2}}{50} \right] + C
\]

Use \( \sin t \) and \( \cos t \) from above.
2. Find the best possible left, right, midpoint, trapezoidal, and Simpson’s approximations to \( \int_{-2}^{0} f(x) \, dx \) given the data in the table below.

<table>
<thead>
<tr>
<th>( x )</th>
<th>-2</th>
<th>-1.5</th>
<th>-1</th>
<th>-0.5</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>2</td>
<td>3</td>
<td>6</td>
<td>10</td>
<td>11</td>
</tr>
</tbody>
</table>

\[ L_4 = (2 + 3 + 6 + 10)(0.5) = 10.5 \quad R_4 = (3 + 6 + 10 + 11)(0.5) = 15 \quad T_4 = 0.5(L_4 + R_4) = 12.75 \]

We cannot compute \( M_4 \), which would require the values of \( f \) at \( x = -1.75, -1.25, -0.75, \) and \(-0.25 \).

Instead, we find \( M_2 : M_2 = (3 + 10)(1) = 13. \)

Finally, \( S_4 = \frac{2M_2 + T_2}{3} \), so we need to compute \( T_2 = \frac{L_2 + R_2}{2} = \frac{(2 + 6)(1) + (6 + 11)(1)}{2} = 12.5. \)

Thus, \( S_4 = \frac{2M_2 + T_2}{3} = \frac{2(13) + 12.5}{3} = \frac{77}{6}. \)

3. If you use numerical integration to estimate \( \int_a^b \ln x \, dx \), how would the following be ordered from least to greatest? \( L_{100}, R_{100}, M_{100}, T_{100}, \int_a^b \ln x \, dx \).

The integrand is increasing and concave down, so we have \( L_{100} < T_{100} < \int_a^b \ln x \, dx < M_{100} < R_{100}. \)

4. Find bounds for each of the following errors if \( I = \int_0^2 e^{-5x} \, dx. \)

(a) \[ |I - R_{100}| \leq \frac{K_1(b - a)^2}{2n} = \frac{5(2 - 0)^2}{2(100)} = \frac{1}{10} \]

\( K_1 = \max \{ |f'(x)| \} \text{ on } [0, 2] = \max \{ 5e^{-5x} \} \text{ on } [0, 2] = 5 \) (occurs at \( x = 0 \))

(b) \[ |I - T_{100}| \leq \frac{K_2(b - a)^3}{12n^2} = \frac{25(2 - 0)^3}{12(100)^2} = \frac{1}{600} \]

\( K_2 = \max \{ |f''(x)| \} \text{ on } [0, 2] = \max \{ 25e^{-5x} \} \text{ on } [0, 2] = 25 \) (occurs at \( x = 0 \))

(c) \[ |I - M_{100}| \leq \frac{K_2(b - a)^3}{24n^2} = \frac{25(2 - 0)^3}{24(100)^2} = \frac{1}{1200} \]

\( K_2 \) = same as in previous part

5. If \( I = \int_0^2 e^{-5x} \, dx \), how many subdivisions are required to obtain a midpoint sum approximation with error of at most \( 1/1,000,000 \)?

From part (c) above, we know that \( |I - M_n| \leq \frac{K_2(b - a)^3}{24n^2} = \frac{25(2 - 0)^3}{24n^2} = \frac{25}{3n^2}. \)

Thus, we want \( \frac{25}{3n^2} \leq \frac{1}{1,000,000} \), which is equivalent to \( \frac{25,000,000}{3} \leq n^2 \).

Taking the square root of each side results in \( \sqrt{25,000,000/3} \leq n. \)

Since \( \sqrt{25,000,000/3} = 2886.751... \), we must at least 2887 subdivisions.
6. Use Euler’s Method with 3 steps to estimate \( y(3/4) \) if \( dy/dx = y - 3 \) and \( y(0) = 1 \).
   Note: Students in the 11:00 and 12:05 sections should omit this problem.

   \[
   \begin{array}{|c|c|c|}
   \hline
   x & y & \frac{dy}{dx} \cdot \Delta x = \Delta y \\
   \hline
   0 & 1 & (-2)(0.25) = -0.5 \\
   0.25 & 0.5 & (-2.5)(0.25) = -0.625 \\
   0.5 & -0.125 & (-3.125)(0.25) = -0.78125 \\
   0.75 & -0.90625 & \\
   \hline
   \end{array}
   \]

7. Write an integral equal to the area between \( y = 2x + 3 \) and \( y = x^2 + 7x - 3 \).
   First, find where the curves intersect.
   \[
   \begin{align*}
   x^2 + 7x - 3 &= 2x + 3 \\
   x^2 + 5x - 6 &= 0 \\
   (x + 6)(x - 1) &= 0 \\
   \Rightarrow x &= -6, 1
   \end{align*}
   \]
   Between \( x = -6 \) and \( x = 1 \), \( y = 2x + 3 \) is above \( y = x^2 + 7x - 3 \). (Plug in \( x = 0 \) to check.) So, the area between them is \( \int_{-6}^{1} [(2x + 3) - (x^2 + 7x - 3)] \, dx \).
   [This equals 343/6.]

8. Compute the arc length of \( y = \sqrt{1 - x^2} \) from \( x = 0 \) to \( x = 1/2 \).
   First, we find \( f'(x) = \frac{1}{2}(1 - x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{1 - x^2}} \).
   Thus, \( [f'(x)]^2 = \frac{x^2}{1 - x^2} \).

   \[
   \int_{0}^{1/2} \sqrt{1 + [f'(x)]^2} \, dx = \int_{0}^{1/2} \sqrt{1 + \frac{x^2}{1 - x^2}} \, dx
   \]
   This is the definition of arc length.
   \[
   = \int_{0}^{1/2} \sqrt{\frac{1}{1 - x^2} + \frac{x^2}{1 - x^2}} \, dx
   \]
   Get a common denominator.
   \[
   = \int_{0}^{1/2} \sqrt{\frac{1}{1 - x^2}} \, dx
   \]
   \[
   = \int_{0}^{1/2} \frac{1}{\sqrt{1 - x^2}} \, dx
   \]
   \[
   = \arcsin x \bigg|_{0}^{1/2}
   \]
   \[
   = \arcsin(1/2) - \arcsin(0)
   \]
   \[
   = \pi/6 - 0
   \]
   \[
   = \pi/6
   \]

9. Consider the region bounded by \( y = 0, x = 2, \) and \( y = x^2 \). Write an integral equal to the volume of the object created when the region is revolved about
   (a) the \( x \)-axis
   Slice vertically into disks.
volume of slice \( \approx \pi r^2 \Delta x \)
\( = \pi y^2 \Delta x \)
\( = \pi (x^2)^2 \Delta x \)
\( = \pi x^4 \Delta x \)

\[ \text{total volume} = \pi \int_0^2 x^4 \, dx \]

(b) the line \( x = 5 \)

Slice horizontally into washers.

volume of slice \( \approx \pi R^2 \Delta y - \pi r^2 \Delta y \)
\( = \pi (5 - x)^2 \Delta y - \pi (3)^2 \Delta y \)
\( = \pi [(5 - \sqrt{y})^2 - 3^2] \Delta y \)

\[ \text{total volume} = \pi \int_0^4 [(5 - \sqrt{y})^2 - 3^2] \, dy \]

10. A spherical tank of radius 8 feet is buried 5 feet below ground and filled to a height of 11 feet with gasoline (42 pounds per cubic foot). Write an integral equal to the work done in pumping all the gasoline to ground level.

\[ \text{volume of slice} \approx \pi r^2 \Delta h = \pi (16h - h^2) \Delta h \]
\[ \text{weight of slice} \approx 42\pi (16h - h^2) \Delta h \]
\[ \text{work to lift slice} \approx 42\pi (16h - h^2) \Delta h (21 - h) \]

\[ \text{total work} = 42\pi \int_0^{11} (16h - h^2)(21 - h) \, dh \]

\[ r^2 + (8 - h)^2 = 8^2 \]
\[ r^2 + 64 - 16h + h^2 = 64 \]
\[ r^2 = 16h - h^2 \]

11. Find the solution to \( \frac{dy}{dx} = \frac{\cos x}{y^2} \) that passes through \((0,2)\). Use separation of variables.

\[ \int y^2 \, dy = \int \cos x \, dx \]
\[ \frac{y^3}{3} = \sin x + C \]
\[ y^3 = 3 \sin x + D \]
\[ y = \sqrt[3]{3 \sin x + D} \]

When \( x = 0 \), we have \( y = 2 \), so \( 2 = \sqrt[3]{3 \sin 0 + D} \), or \( 2 = \sqrt[3]{D} \). Thus, \( D = 8 \).

Therefore, the solution is \( y = \sqrt[3]{3 \sin x + 8} \).
12. The probability density function (pdf) of the weights of newborn toads in a certain pond is given by \( f(x) = \frac{k}{(x+1)^4} \), where \( x \) is the weight (in ounces). Note that the domain is \( x \geq 0 \) since no toad can have a negative weight.

(a) What must be the value of \( k \)?

We know that the total area under any pdf must be 1 (because it must account for 100% of events.)

\[
\int_0^\infty \frac{k}{(x+1)^4} \, dx = \lim_{t \to \infty} \int_0^t \frac{k}{(x+1)^4} \, dx \\
= \lim_{t \to \infty} \left[ \frac{k(x+1)^{-3}}{-3} \right]_0^t \\
= \lim_{t \to \infty} \frac{k}{-3} - \frac{k}{-3(0+1)^3} \\
= 0 - \frac{k}{-3} \\
= \frac{k}{3}
\]

So, we have \( k/3 = 1 \) or \( k = 3 \).

(b) What fraction of the newborn toads weigh more than one ounce?

\[
\int_1^\infty \frac{3}{(x+1)^4} \, dx = \lim_{t \to \infty} \int_1^t \frac{3}{(x+1)^4} \, dx \\
= \lim_{t \to \infty} \left[ -3(x+1)^3 \right]_1^t \\
= \lim_{t \to \infty} \frac{1}{-1(t+1)^3} - \frac{1}{-1(1+1)^3} \\
= 0 - \frac{1}{-8} \\
= \frac{1}{8}
\]

Note that we could instead have computed \( 1 - \int_0^1 \frac{3}{(x+1)^4} \, dx \) and gotten the same answer.