Some of the following may be useful.

$$\begin{bmatrix}
27 & 9 & 1 & 47 \\
8 & 4 & 1 & 10 \\
-1 & 1 & 1 & -5 \\
-8 & 4 & 1 & -38 \\
\end{bmatrix} \sim \begin{bmatrix}
1 & 0 & 0 & 3 \\
0 & 1 & 0 & -4 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 \\
\end{bmatrix} \sim \begin{bmatrix}
47 & 27 & 9 & 1 \\
10 & 8 & 4 & 1 \\
-5 & -1 & 1 & 1 \\
-38 & -8 & 4 & 1 \\
\end{bmatrix} \sim \begin{bmatrix}
1 & 0 & 0 & -3 \\
0 & 1 & 0 & 4 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}$$

$$\begin{bmatrix}
114 & 20 & 55 \\
20 & 4 & 14 \\
14 & 4 & 20 \\
70 & 14 & 55 \\
\end{bmatrix} \sim \begin{bmatrix}
1 & 0 & -15/14 \\
0 & 1 & 62/7 \\
1 & 0 & -5/7 \\
0 & 1 & 15/2 \\
\end{bmatrix} \sim \begin{bmatrix}
70 & 14 & 55 \\
14 & 4 & 20 \\
1 & 0 & -5/7 \\
0 & 1 & 15/2 \\
\end{bmatrix} \sim \begin{bmatrix}
1 & 0 & -5/7 \\
0 & 1 & 15/2 \\
\end{bmatrix}$$

$$\begin{bmatrix}
1 & 0 & 4 & 0 & 2 & 1 & 0 & 0 & -1 \\
0 & 1 & -3 & 0 & 5 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 3 & 2 & 1 & -3 \\
0 & 0 & 0 & 0 & -11 & -4 & -2 & 12 \\
\end{bmatrix} \sim \begin{bmatrix}
6 & 0 & 24 & 2 & 14 & 1 & 0 & 0 & 0 \\
0 & 1 & -3 & 0 & 5 & 0 & 1 & 0 & 0 \\
-3 & -2 & -6 & 1 & -15 & 0 & 0 & 1 & 0 \\
5 & 0 & 20 & 2 & 12 & 0 & 0 & 0 & 1 \\
\end{bmatrix}$$
1. There is a cubic polynomial of the form $Ax^3 + Bx^2 + C$ which contains the four points $(3, 47)$, $(2, 10)$, $(-1, -5)$ and $(-2, -38)$.

1A. Set up the system of equations you need to solve in order to find $A$, $B$ and $C$.

1B. Solve the system, and find the polynomial. This is not a least squares problem!
2. The points $(1, 8)$, $(2, 5)$, $(4, 4)$ and $(7, 3)$ do not lie on any one line. Find $m$ and $b$ so that $y = mx + b$ is the least squares line that best fits these four points. Show all your work.
3. Suppose $W$ is the column space of \( A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} \) and let \( z = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \).

3A. What equation(s) does \( z \) have to satisfy in order for \( z \) to be in \( W^\perp \)?

3B. What matrix equation represents the answer to (3A)?

3C. Use (3B) to find a basis for \( W^\perp \).

3D. Let \( S \) be the set of those two column vectors in \( A \). Note that \( S \) is orthogonal. Produce an orthonormal set of vectors which also spans \( W \).

3E. Use \( S \) and our dot “product formula” to find the projection \( w \) of \( y = \begin{bmatrix} 5 \\ 3 \\ 4 \\ 2 \end{bmatrix} \) onto \( W \).

3F. What is the distance from \( w \) to \( y \)?
4. Let \( A = \begin{bmatrix} 3 & -8 & 6 \\ 3 & -7 & 3 \\ 3 & -4 & 0 \end{bmatrix} \); then \( A \) is diagonalizable. Find \( P \) and \( D \) with the “right properties”.

**Facts:** One eigenvalue for \( A \) is 2 and one eigenvector for \( A \) is \( \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix} \).

*Show all your steps!*
5. Define $T: \mathbb{P}_4 \to \mathbb{P}_4$ by $T(p(x)) = p'(x)$; then $T$ is a linear transformation (LT).

5A. Find (describe) the kernel of this LT.

5B. Is $1 + x + x^2 + x^3$ in the image of $T$? Explain.

5C. Is $1 + 5x^4$ in the image of $T$? Explain.
6. Suppose that \( S = \{ u_1, u_2, \ldots, u_p \} \) is a set of vectors all having length 2, and that \( W = \text{span}(S) \). Suppose furthermore that \( S \cup \{ b \} = \{ u_1, u_2, \ldots, u_p, b \} \) is orthogonal, and \( b \) is not equal to any of the \( u \)'s.

6A. Show that \( b \) is perpendicular to any vector in \( W \), and hence is in \( W^\perp \).

6B. Is \( S \) a basis for \( W \)? Explain.

6C. Can \( b \) be a linear combination of the members of \( S \)? Explain any possibilities.
7. Let \( G = \begin{bmatrix} 6 & 0 & 24 & 2 & 14 \\ 0 & 1 & -3 & 0 & 5 \\ -3 & -2 & -6 & 1 & -15 \\ 5 & 0 & 20 & 2 & 12 \end{bmatrix} \) and let \( \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \).

7A. Is every such \( \mathbf{b} \) in the column space of \( G \)? If not, describe any conditions that must be met.

7B. What is the dimension of the column space of \( G \)?

7C. Find all solutions of \( G \mathbf{x} = \begin{bmatrix} 28 \\ -15 \\ 2 \\ 21 \end{bmatrix} \). Express your answer in terms of a particular solution and the solutions of the corresponding homogeneous equation.

7D. What is the dimension of the null space of \( G \)?

7E. Find a basis for the null space of \( G \).

7F. Find a basis for the row space of \( G \).
8. For each of the following vector spaces, give two obvious bases with no common vectors, or explain why there is no basis. Also then give the dimension of the vector space:

8A. $P_3$

8B. $\{0\}$

8C. $\mathbb{R}^4$

8D. The space $S$ of all sequences $(s_1, s_2, s_3, \ldots)$ of real numbers, with addition and scalar multiplication as discussed in class.

9. Suppose $B \in M_{4 \times 6}$ and its RREF form has at least one row of zeros.

9A. What are the maximum and minimum possible dimensions for $\text{Nul}(B)$? MAX: MIN:

9B. What are the maximum and minimum possible values for the rank of $B$? MAX: MIN:
Suppose \( D = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \) and \( b = \begin{bmatrix} 1 \\ 3 \\ 8 \\ 2 \end{bmatrix} \); then the least squares solution to \( Dx = b \) is

\[
\begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}
\]

where \( x_3 \) is free.

10A. Do the columns of \( D \) form orthogonal set? Explain.

10B. Find the projection of \( b \) onto the column space of \( D \). Show your work.