**DO NOT WRITE HERE!**

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Read the questions CAREFULLY.

Show your work in the space provided.

Make clear what your answers are.

BE NEAT.

Good Luck!
1. Let $F$ be the vector space of continuous functions $f : \mathbb{R} \to \mathbb{R}$ that we have been using in class. Let $H = \{ f \in F \mid f$ has the same $y$-coordinate at $x = -1$ as it does at $x = 1 \}$, so a vector $u$ belongs to $H$ iff $u$ is a function $f$ satisfying $f(-1) = f(1)$.

   (1A) Give an example of a non-constant function $f$ belonging to $H$.

   (1B) It’s true that $H$ a subspace of $F$. For just the first TWO of the three conditions given in the three parts of the definition of a subspace (in the order we’ve always talked about them), prove that $H$ satisfies that condition.

   1Bi) proof that the first condition, or part, of the subspace definition holds:

   1Bii) proof that the second condition of the subspace definition holds:
2. Let $S$ be the vector space of all sequences $s = (s_1, s_2, s_3, \ldots)$ of real numbers that we’ve discussed in class, and $F$ be the vector space of continuous functions $f : \mathbb{R} \to \mathbb{R}$. Define $T : F \to S$ by

$$T(f) = (f(1) + 2, f(2) + 3, f(3) + 4, f(4) + 5, \ldots).$$

For example, if $f(x) = x^2$, then the second term in sequence $T(f)$ is $f(2) + 3 = 2^2 + 3 = 4 + 3 = 7$. The first two terms of $T(f)$ are then $(3, 7, \ldots)$.

(2A) So, if $f(x) = x^2$, what are the first five terms of the sequence $T(f)$?

(2B) Find $T(g)$ where $g(x) = x^3$. (Give the first five terms).

(2C) Find $T(5f) = T(5x^2)$ through the first five terms.

(2D) Is $T$ a linear transformation? For each of the two parts of the definition of a linear transformation, either prove that $T$ satisfies that part, or show it does not by giving a specific counterexample. I’d recommend considering the examples you’ve worked on in 2A–2C!

(2Di) (part one of the LT definition: your proof or counterexample):

(2Dii) (part two of the LT definition: your proof or counterexample):
3. Let \( A = \begin{bmatrix} 1 & 1 & -5 \\ 1 & 2 & 4 \\ 1 & -3 & 1 \end{bmatrix} \); let \( \mathbf{v}_1, \mathbf{v}_2, \) and \( \mathbf{v}_3 \) be the column vectors of \( A \), and let \( S = \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \} \).

(3A). Explain why \( A\mathbf{x} = \mathbf{b} \) has a solution \( \mathbf{x} \) for any \( \mathbf{b} \in \mathbb{R}^3 \).

(3B). Explain why the set \( S \) is linearly independent.

(3C). Parts 3A and 3B say that \( S \) is a basis for \( \mathbb{R}^3 \). Show that it is an orthogonal basis.

(3D.) Since \( S \) is an orthogonal basis of \( \mathbb{R}^3 \), there’s a formula that uses dot-products to find the weights needed to express a vector \( \mathbf{u} \in \mathbb{R}^3 \) as a LC of the members of \( S \). What is that formula?

(3E). Use the formula in 3D to express \( \mathbf{u} = \begin{bmatrix} -333 \\ 493 \\ -109 \end{bmatrix} \) as a linear combination of the vectors in \( S \). Show all your work. Express all weights as fractions in lowest terms.
4. Let \( A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 3 & 1 & 3 \\ 5 & 2 & 4 \end{bmatrix} \) and \( y = \begin{bmatrix} 9 \\ 18 \\ 28 \\ 4 \end{bmatrix} \).

(4A) It turns out that \( y \) is not in the column space of \( A \), but you do not need to check this. Find the least squares solution of \( Ax = y \). Show all your work.

(4B) Find the projection of \( y \) onto the column space of \( A \). Note well the columns are not orthogonal!

(4B) Find the distance from \( y \) to \( \text{Col}(A) \). Show all your work.
5. In this problem, in your answers and work, write all repeating decimal numbers to 4 places after the decimal point (not fractions). For example, write 1/3 as 0.3333... But maintain complete precision in the calculator itself.

There is no parabola passing through the points (2, 3) (3, 5) (4, 3 1/6) (6, 6 1/3); you do not need to verify this. [NOTE that 3 1/6 means 19/6 = 3.166666... and 6 1/3 = 19/3 = 6.333333... — enter such numbers into your calculator as fractions (eg, 19/6 and 19/3) to maintain maximum precision].

(5A) Find the parabola \( y = \beta_2 x^2 + \beta_1 x + \beta_0 \) that is the “best-fit” parabola for these points. Show all your work, including all matrices and vectors involved in solving this problem. Remember: write just 4 places for repeating decimal expansions.

(5B) What are the predicted values, that is, the \( y \)-coordinates of this best-fit parabola at \( x = 2, 3, 4 \) and 6, respectively?

(5C) What is the sum-of-the-squares (“SOS”) of the residuals for the predicted values found in (5B)? Show all your computations.

(5D) A polynomial whose coefficients are “sort of close” to the best-fit coefficients is \( y = 0.2x^3 - 0.5x + 4 \). If this polynomial were used to find the predicted values, what would they be?

(5E) What is the SOS of the residuals for the predicted values in 5D?
6. Let \( M = \begin{bmatrix} -15 & 22 & -11 \\ 44 & -92 & 44 \\ 110 & -220 & 106 \end{bmatrix} \)

6a) Let \( \mathbf{a} = \begin{bmatrix} -1 \\ 4 \\ 10 \end{bmatrix} \). Find \( Ma \). Is \( \mathbf{a} \) an eigenvector for \( M \)? If so, what’s the corresponding eigenvalue?

6b) It turns out that \(-4\) is an eigenvalue for \( M \). Find a basis for the corresponding eigenspace.

6c) Find the characteristic polynomial of \( M \) (this should be easy based on the previous two parts).

6d) Find, if possible, \( P \) and \( D \) for which \( M = PDP^{-1} \), and \( D \) is a diagonal matrix whose entries are eigenvalues of \( M \) and the corresponding columns of \( P \) are eigenvectors corresponding to those eigenvalues.

6e) Use your calculator to find \( P^{-1} \).
7. Let \( A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 3 & 1 & 3 \\ 5 & 2 & 4 \end{bmatrix} \) (same as in problem 4). Let \( R \) be the rref of \( A \). Find each of the following. Show any relevant matrices you used in your computations:

7A) Find a basis for \( \text{col}(A) \). Call this basis \( B \).

7B) Find a basis for \( \text{col}(R) \).

7C) Find a basis for \( \text{row}(A) \).
Problem 7, continued:

7D) Find a basis for \( \text{row}(R) \).

7E) Find a basis for \( \text{null}(A) \).

7F) Find a basis for \( \text{col}(A)^\perp \).
8. Again, let \( A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 3 & 1 & 3 \\ 5 & 2 & 4 \end{bmatrix} \) (same as in problem 4 and 7).

8A) Let \( b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \). Find all conditions on \( b_1, \ldots, b_4 \) which guarantee that \( Ax = b \) has a solution \( x \)

8B) Find \( b_1 \) and \( b_2 \) for which \( w = \begin{bmatrix} b_1 \\ b_2 \\ 13 \\ 21 \end{bmatrix} \) is in \( \text{col}(A) \).

8C) Find \( [w]_B \) (see 7A, where \( B \) is defined, and 8B).

9. Suppose the determinant of some 4x4 matrix \( M \) is 5. Next to each of the following matrices, write its determinant.

\[
\begin{align*}
M^3 & \quad 3M \\
-M & \quad 4M + 3M + 2M + M
\end{align*}
\]