1. (12 pts.) Recall that the Taylor series for $\arctan x$ centered at $x_0 = 0$ is

$$\arctan x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1} \quad \text{on } [-1, 1].$$

(a) Find the first four non-zero terms of the Taylor series for $f(x) = x^3 \arctan(-x^2)$ centered at $x_0 = 0$.

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

$$\arctan(-x^2) = (-x^2) - \frac{(-x^2)^3}{3} + \frac{(-x^2)^5}{5} - \frac{(-x^2)^7}{7} + \cdots = -x^2 + \frac{x^6}{3} - \frac{x^{10}}{5} + \frac{x^{12}}{7} - \cdots$$

$$x^3 \arctan(-x^2) = x^3 \left(-x^2 + \frac{x^6}{3} - \frac{x^{10}}{5} + \frac{x^{12}}{7} - \cdots\right)$$

$$= -x^5 + \frac{x^9}{3} - \frac{x^{13}}{5} + \frac{x^{15}}{7} - \cdots$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^k x^{4k+1}}{2k-1}$$

(b) Compute a partial sum for the series above to estimate $f(1)$ with error less than $\frac{1}{10}$.

$$f(1) = \sum_{k=1}^{\infty} \frac{(-1)^k}{2k-1}$$ is an alternating series which converges by AST.

Since $C_k = \frac{1}{2k-1} > 0$ as $k \to \infty$ and $0 < C_k < C_{k+1}$ for all $k$ since $0 < \frac{1}{2k+1} < \frac{1}{2k-1}$, the error in using the partial sum $S_N$ to estimate $S$ is then

The error is given by $|S_N - S| < C_{N+1}$.

$$C_{N+1} = \frac{1}{2(N+1)-1} = \frac{1}{2N+1} < \frac{1}{10} \quad \Leftrightarrow \quad 2N+1 > 10 \quad \Leftrightarrow \quad N > 5$$

$$S_5 = -1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} = \frac{-263}{315} \approx -0.835.$$
2. (7 pts.) A cistern has the shape of the lower half of a sphere of radius 5 ft and is half full of water. *Set up, but do not evaluate*, the integral which represents the work required to pump all of the water to a point 4 feet above the top of the cistern. Recall: the weight of water is 62.4 lb/ft$^3$.

\[
\int_{\frac{3}{5}}^{2.5} 62.4 \pi (25-y^2) (y-y) \, dy
\]

Note: $x^2 + y^2 = 25 \implies x^2 = 25 - y^2$

\[
V_{slab} = \pi x^2 \, dy = \pi (25 - y^2) \, dy
\]

Distance to move slab = $4 - y$

Weight slab $\approx 62.4 \pi (25 - y^2) \, dy$

3. (6 pts.) Use separation of variables to solve the following initial value problem. Be sure to use the initial condition to determine the exact value of any constant you introduce.

\[
\frac{dy}{dt} = 2t \sqrt{1 - y^2} \quad \text{with} \quad y(0) = \frac{1}{2}
\]

\[
\frac{dy}{\sqrt{1 - y^2}} = 2t \, dt
\]

Thus $\arcsin y = t^2 + \frac{\pi}{6}$

\[
\arcsin \frac{1}{2} = \frac{\pi}{6}
\]

$\arcsin y = t^2 + \frac{\pi}{6}$

\[
y = \sin \left( t^2 + \frac{\pi}{6} \right)
\]

$\Rightarrow c = \frac{\pi}{6}$

\[
c = \frac{\pi}{6}
\]
4. (7 pts. each) Evaluate the following integrals. Give exact values please.

(a) \[ \int_{0}^{\pi/3} x \sec^2 x \, dx \]

\[ \begin{align*}
\text{Let } u &= x, & du &= dx \\
\text{dv} &= \sec^2 x \, dx, & v &= \tan x
\end{align*} \]

\[ \int_{0}^{\pi/3} x \sec^2 x \, dx = \left[ \tan^2 x \right]_{0}^{\pi/3} = \frac{\pi}{3} \]

\[ \int_{0}^{\pi/3} x \sec^2 x \, dx = \frac{\pi}{3} \cdot \tan \left( \frac{\pi}{3} \right) - \left( 0 + \ln \left| \cos x \right| \right)_{0}^{\pi/3} \]

\[ = \frac{\pi}{3} \cdot \sqrt{3} - \ln 1 = \frac{\pi}{3} - \ln 2 \]

(b) \[ \int_{0}^{12} \frac{dx}{\sqrt{25 + x^2}} \]

\[ \begin{align*}
\text{Let } x &= 5 \tan \theta, & dx &= 5 \sec^2 \theta \, d\theta
\end{align*} \]

\[ \int_{0}^{12} \frac{dx}{\sqrt{25 + x^2}} = \int_{0}^{\pi/2} \frac{5 \sec^2 \theta \, d\theta}{\sqrt{25 + 25 \tan^2 \theta}} \]

\[ = \int_{0}^{\pi/2} \frac{5 \sec^2 \theta}{5 \sec \theta} \, d\theta = \int_{0}^{\pi/2} \sec \theta \, d\theta \]

\[ = \left( \ln \left| \sec \theta + \tan \theta \right| \right)_{0}^{\pi/2} \]

\[ = \left( \ln \left| \frac{25 + x^2}{5} + \frac{x}{5} \right| \right)_{0}^{12} = \ln \left| \frac{25 + 144 + 12}{5} \right| - \ln \left| \frac{25 + 0}{5} + \frac{0}{5} \right| \]

\[ = \ln \frac{25}{5} - \ln 1 = \ln 5 \]
5. (10 pts.) Consider \( \sum_{k=1}^{\infty} \frac{8}{(4k-3)(4k+1)} \).

(a) Use partial fractions to rewrite \( a_k = \frac{8}{(4k-3)(4k+1)} \).

\[
\frac{8}{(4k-3)(4k+1)} = \frac{A}{4k-3} + \frac{B}{4k+1}
\]

\[
8 = A(4k+1) + B(4k-3)
\]

\[
\begin{align*}
\text{at } k = \frac{3}{4} : & \quad 8 = 4A \quad \Rightarrow \quad A = 2 \\
\text{at } k = \frac{1}{4} : & \quad 8 = -4B \quad \Rightarrow \quad B = -2
\end{align*}
\]

Therefore:

\[
\frac{8}{(4k-3)(4k+1)} = \frac{2}{4k-3} - \frac{2}{4k+1}
\]

(b) Find the first 3 terms for the sequence of partial sums for \( \sum_{k=1}^{\infty} a_k \). Be sure to use your answer for \( a_k \) from part (a). Give exact values, no decimals—this will help you to identify the pattern and general formula required for part (c) below.

\[
S_1 = \frac{2}{1} - \frac{2}{5}
\]

\[
S_2 = \left(2 - \frac{2}{5}\right) + \left(\frac{2}{5} - \frac{2}{9}\right) = 2 - \frac{2}{9}
\]

\[
S_3 = \left(2 - \frac{2}{5}\right) + \left(\frac{2}{5} - \frac{2}{9}\right) + \left(\frac{2}{9} - \frac{2}{13}\right) = 2 - \frac{2}{13}
\]

(c) Use part (b) to find a formula for the \( n^{th} \) partial sum, \( S_n \). Then find \( \lim_{n \to \infty} S_n \).

\[
S_n = 2 - \frac{2}{4n+1} \quad \Rightarrow \quad 2 \quad \text{as} \quad n \to \infty
\]

(d) What does this tell you about \( \sum_{k=1}^{\infty} \frac{8}{(4k-3)(4k+1)} \)?

\[
S_n \to 2 \quad \Rightarrow \quad \sum_{k=1}^{\infty} \frac{8}{(4k-3)(4k+1)} = 2
\]
6. (12 pts.) Consider the following functions and their graphs:

\[ f(x) = \frac{1}{1 + x^2} \quad g(x) = e^{-2(x-2)^2} \quad h(x) = \frac{e^{2+\cos x}}{x^2} \]

Each of the improper integrals below converge.

\[ \int_1^\infty f(x) \, dx, \quad \int_1^\infty g(x) \, dx, \quad \text{and} \quad \int_1^\infty h(x) \, dx \]

(a) Can you use the integral test to determine if \( \sum_{k=1}^{\infty} \frac{1}{1 + k^2} \) converges? Explain your answer.

Yes, since \( f \) is positive, continuous, and decreasing for \([1, \infty)\),
with \( f(k) = \frac{1}{1 + k^2} \), then by the integral test since \( \int_1^\infty f(x) \, dx \) converges, it follows that \( \sum_{k=1}^{\infty} \frac{1}{1 + k^2} \) also converges.

(b) Can you use the integral test to determine if \( \sum_{k=1}^{\infty} e^{-2(k-2)^2} \) converges? Explain your answer.

Not directly ... note \( g \) is continuous, positive, and decreasing on \([2, \infty)\),
with \( g(k) = e^{-2(k-2)^2} \), so since \( \int_2^\infty g(x) \, dx \) converges then by the integral test we know \( \sum_{k=2}^{\infty} e^{-2(k-2)^2} \) converges.
And since \( \sum_{k=1}^{\infty} e^{-2(k-2)^2} = e^2 + \sum_{k=2}^{\infty} e^{-2(k-2)^2} \), then \( \sum_{k=1}^{\infty} e^{-2(k-2)^2} \) must also converge.

(c) Can you use the integral test to determine if \( \sum_{k=1}^{\infty} \frac{e^{2+\cos k}}{k^2} \) converges? Explain your answer.

No, the integral test can’t be used here since \( h \) is not decreasing on \([b, \infty)\) for any \( b \). If we want to show \( \sum_{k=1}^{\infty} \frac{e^{2+\cos k}}{k} \) converges we need to use a different test.
7. (13 pts.) Find the interval of convergence for the series \( \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k2^k} (x+3)^k \). Be sure to check endpoints.

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(x+3)^{k+1}}{(x+3)^k \cdot \frac{k+1}{k+2}} \cdot \frac{k}{2^k} \cdot \frac{2^{k+1}}{2^k} \right| = \lim_{n \to \infty} \frac{|x+3|}{2} \cdot \frac{k}{k+1} = \frac{1}{2} |x+3| \quad \text{since} \quad \frac{k}{k+1} \to 1 \quad \text{as} \quad k \to \infty
\]

For convergence we require \( \frac{1}{2} |x+3| < 1 \iff |x+3| < 2 \iff -2 < x + 3 < 2 \iff -5 < x < -1 \)

If \( x = -5 \):

\[
\sum_{k=1}^{\infty} \frac{(-1)^{k+1}(-5+3)^k}{k2^k} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(-2)^k}{k2^k} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(-1)^k}{k} = \sum_{k=1}^{\infty} \frac{(-1)^{2k+1}}{k} \quad \text{diverges harmonic series}
\]

If \( x = -1 \):

\[
\sum_{k=1}^{\infty} \frac{(-1)^{k+1}(-1+3)^k}{k2^k} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}2k}{k2^k} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \quad \text{alternating harmonic series converges by AST}
\]

Interval of convergence: \( -5 < x < -1 \)

8. (7 pts.) The shelf life, \( x \) (in hours), of a certain perishable packaged food is a random variable whose probability density function is given by \( p(x) = \frac{20,000}{(x+100)^3} \) for \( x > 0 \). Find the probability that one of these packages will have a shelf life of more than 200 hours.

\[
P(x > 200) = \int_{200}^{\infty} \frac{20,000}{(x+100)^3} \, dx = \lim_{b \to \infty} \int_{200}^{b} \frac{20,000}{(x+100)^3} \, dx = \lim_{b \to \infty} \int_{200}^{b} \frac{20,000}{u^3} \, du\quad \text{let} \quad u = x+100, \quad du = dx
\]

\[
= \lim_{b \to \infty} \left[ \frac{20,000}{-2} u^{-2} \right]_{200}^{b} = \lim_{b \to \infty} \frac{-10000}{(b+100)^2} + \frac{10000}{(200)^2} = \frac{-10000}{90000} = \frac{1}{9}
\]
(a) Suppose $f$ is increasing, but concave down. If a trapezoid sum is used to approximate $\int_a^b f(x) \, dx$, then the approximation will be an overestimate / underestimate (circle one) because...

\[ f \text{ concave down means that each trapezoid will sit under the curve thereby producing an underestimate.} \]

(b) Taylor polynomials are designed to have certain features in common with the function they are approximating. If $P_2(x)$ is the second order Taylor polynomial approximating $f(x)$ for $x$ near $a$, then the graphs of $P_2(x)$ and $f(x)$ will have the following features in common...

\[ f(a) = P_2(a) \quad \text{graphs of } f \text{ and } P_2 \text{ have same } y\text{-value at } x=a \]
\[ f'(a) = P_2'(a) \quad \text{graphs of } f \text{ and } P_2 \text{ have same slope at } x=a \]
\[ f''(a) = P_2''(a) \quad \text{graphs of } f \text{ and } P_2 \text{ have same concavity at } x=a \]

(c) The series $\frac{6}{5} - \frac{18}{25} + \frac{54}{125} - \frac{162}{625} + \cdots$ may be written using summation notation as $\sum_{k=0}^{\infty} \frac{6}{5} \left( -\frac{3}{5} \right)^k$. This series converges / diverges (circle one) because ...

\[ \left| r \right| = \left| \frac{3}{5} \right| < 1 \]

\[ \Rightarrow \frac{6}{5} \left( 1 - \frac{3}{5} + \frac{9}{25} - \cdots \right) = \sum_{k=0}^{\infty} \frac{6}{5} \left( \frac{3}{5} \right)^k \]
10. (7 pts.) Consider the region bounded by \( y + x = 2 \) and \( y + 2x^2 = 3 \). Set up, but do not evaluate, the integral that represents the volume of the solid found by revolving this region about the line \( y = 4 \).

Intersection points:
\[-x + 2 = -2x^2 + 3\]
\[-2x^2 - x - 1 = 0\]
\[(2x + 1)(x - 1) = 0\]
\[x = -\frac{1}{2}, 1\]

Outer Radius: \( R = 4 - (-x+2) = 2 + x \)

Inner Radius: \( r = 4 - (-2x^2 + 3) = 1 + 2x^2 \)

\[V = \pi \int_{-\frac{1}{2}}^{1} \left( (2+x)^2 - (1+2x^2)^2 \right) \, dx\]

**BONUS: DO ONE OF THE FOLLOWING:**

- Refer to Problem #1 on page 1: What is \( f^{(105)}(0) \)? Justify your answer.

- Use a comparison to determine if \( \sum_{k=1}^{\infty} \frac{e^{2+\cos k}}{k^2} \) converges.

Note: \( \frac{\cos x}{x} \leq 1 \) for all \( x \) in \( [0, \pi] \).