1. Use a second-degree Taylor polynomial to estimate $\sqrt[3]{28}$.

   We choose $f(x) = \sqrt[3]{x}$ and $x_0 = 27$ because 27 is the perfect cube closest to 28.

   \[
   f(x) = x^{1/3} \quad \quad \quad \quad f(27) = 3
   \]

   \[
   f'(x) = \frac{1}{3} x^{-2/3} = \frac{1}{3x^{2/3}} \quad \quad \quad \quad f'(27) = \frac{1}{27}
   \]

   \[
   f''(x) = -\frac{2}{9} x^{-5/3} = -\frac{2}{9x^{5/3}} \quad \quad \quad \quad f''(27) = -\frac{2}{2187}
   \]

   Now plug in to the Taylor polynomial formula with $x_0 = 27$.

   \[
   P_2(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 = 3 + \frac{1}{27}(x - 27) - \frac{1}{2187}(x - 27)^2
   \]

   Finally, evaluate at $x = 28$.

   \[
   \sqrt[3]{28} \approx P_2(28) = 3 + \frac{1}{27}(28 - 27) - \frac{1}{2187}(28 - 27)^2 = \frac{6641}{2187} \approx 3.0365797
   \]

2. What is the largest possible error that could have occurred in your previous estimate?

   We know that $|f(x) - P_n(x)| \leq \frac{K_{n+1}}{(n+1)!}|x - x_0|^{n+1}$.

   In this case, $n = 2$, $x_0 = 27$, and $x = 28$.

   \[
   K_3 = \text{max of } |f'''(x)| \text{ on } [27, 28] = \text{max of } \left| \frac{10}{27x^{8/3}} \right| \text{ on } [27, 28] = \frac{10}{27 \cdot 27^{8/3}} = \frac{10}{177147}
   \]

   Putting this all together, we have $|f(x) - P_2(x)| \leq \frac{10}{3!} |28 - 27|^3 = \frac{5}{531441} \approx 0.0000094$.

3. Use a comparison to show whether each of the following converges or diverges. If an integral converges, give a good upper bound for its value.

   (a) $\int_1^\infty \frac{7 + 5 \sin x}{x^2} \, dx$

   For all $x \geq 1$, we have $0 \leq \frac{7 + 5 \sin x}{x^2} \leq \frac{7 + 5(1)}{x^2} = 12 \frac{1}{x^2}$ because the maximum of $\sin x$ is 1.

   \[
   12 \int_1^\infty \frac{dx}{x^2} = 12 \lim_{t \to \infty} \int_1^t \frac{dx}{x^2} = 12 \lim_{t \to \infty} -\frac{1}{x} \bigg|_1^t = 12 \lim_{t \to \infty} \left[ -\frac{1}{t} - \frac{-1}{1} \right] = 12[0 - (-1)] = 12
   \]

   Therefore, the original integral in question must converge to a value less than 12.

   (b) $\int_1^\infty \frac{1 + 3x^2 + 2x^3}{\sqrt{10x^{12} + 17x^{10}}} \, dx$

   For $x \geq 1$, we have $\frac{1 + 3x^2 + 2x^3}{\sqrt{10x^{12} + 17x^{10}}} \geq \frac{2x^3}{\sqrt{10x^{12} + 17x^{10}}} \geq 0$. (We've made the numerator smaller and the denominator larger, so the new fraction is smaller.)
But \[
\frac{2x^3}{\sqrt{10x^2 + 12x^2}} = \frac{2x^3}{\sqrt{27x^2}} = \frac{2x^3}{3x^2} = \frac{2}{3} x
\] and we know that \[\frac{2}{3} \int_1^\infty \frac{dx}{x}\] diverges (compute for yourself or notice that \(p = 1\)). Therefore the original integral must also diverge.

4. Decide if each of the following sequences \(\{a_k\}_{k=1}^\infty\) converges or diverges. If a sequence converges, compute its limit.

(a) \(a_k = 3 + \frac{1}{10^k}\) 
Terms are 3.1, 3.01, 3.001, 3.0001, ....
\[\lim_{k \to \infty} \left(3 + \frac{1}{10^k}\right) = 3,\] so the sequence converges to 3.

(b) \(a_k = (-1)^k\) 
Terms are \(-1, 1, -1, 1, \ldots\)
\[\lim_{k \to \infty} (-1)^k\] doesn’t exist, so the sequence diverges.

(c) \(a_k = \frac{3 + 5k}{7 + 2k}\) 
Terms are 8/9, 13/11, 18/13, 23/15, ....
\[\lim_{k \to \infty} \frac{3 + 5k}{7 + 2k} = \frac{5}{2}\] (by L’Hopital’s Rule or by inspection), so the sequence converges to \(\frac{5}{2}\).

5. Decide if each of the following series converges or diverges. If a series converges, find its value.

(a) \(3.1 + 3.01 + 3.001 + 3.0001 + \ldots\)
\[\lim_{k \to \infty} a_k = 3 \neq 0,\] so the series diverges by the nth Term Test. (We keep adding 3’s forever.)
[Compare this with the first sequence of the previous problem.]

(b) \(1 + 1/2 + 1/3 + 1/4 + \ldots\)
This is the famous Harmonic Series, which diverges even though the terms do approach 0. We can use the Integral Test: \[\int_1^{\infty} \frac{1}{x} \, dx\] diverges, which means that \(\sum_{k=1}^{\infty} \frac{1}{k}\) must diverge too.

(c) \(5 - 5/3 + 5/9 - 5/27 + \ldots\)
This is a geometric series with \(r = -\frac{1}{3}\), so it converges to \[\frac{a}{1 - r} = \frac{5}{1 - (-1/3)} = \frac{15}{4}\]

6. Decide if each of the following series converges or diverges. If a series converges, find upper and lower bounds for its value.

(a) \[\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt[k]{k+1}}\] [Alternating Series Test]
The terms of this series alternate in sign.
And, \[\frac{1}{\sqrt{2}} \geq \frac{1}{\sqrt{3}} \geq \frac{1}{\sqrt{4}} \geq \ldots \geq 0.\]
And, \[\lim_{k \to \infty} \frac{1}{\sqrt[k]{k+1}} = 0.\]
Therefore, by the Alternating Series Test, the series must converge.
We know that any two consecutive partial sums will provide upper and lower bounds:
lower bound = \(S_1 = -\frac{1}{\sqrt{2}}\) \quad upper bound = \(S_2 = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}}\)

[To get better bounds, use later partial sums, such as \(S_5\) and \(S_6\).]
(b) $\sum_{k=1}^{\infty} \frac{(2k)!}{3k(k!)^2}$  

$$\lim_{k \to \infty} \frac{a_{k+1}}{a_k} = \frac{\frac{[2(k+1)]!}{3^{k+1}[(k+1)!]^2}}{\frac{(2k)!}{3^k(k!)^2}}$$

$$= \lim_{k \to \infty} \frac{(2k + 2)!}{(2k)!} \cdot \frac{3^k}{3^{k+1}} \cdot \frac{(k!)^2}{[(k+1)!]^2} \cdot \frac{1}{1} \cdot \frac{1}{3} \cdot \frac{[(k+1)!]^2}{(k!)^2}$$

$$= \frac{4k^2 + 6k + 2}{3k^2 + 6k + 3}$$

Use L’Hospital or divide each term by $k^2$.

$$= \frac{4}{3}$$

Since the limit of the ratio is greater than 1, the series diverges.

(c) $\sum_{k=1}^{\infty} \left( \frac{1}{100} + \frac{1}{k^5} \right)$  

$$\lim_{k \to \infty} \left( \frac{1}{100} + \frac{1}{k^5} \right) = \frac{1}{100} \neq 0,$$  

so, by the nth Term Test, the series diverges.

(d) $\sum_{k=1}^{\infty} \frac{\sqrt{9k^8 + 5k^6}}{12k^5 + 3}$  

[Comparison Test]

For $k \geq 1$, we have $\frac{\sqrt{9k^8 + 5k^6}}{12k^5 + 3} > \frac{\sqrt{9k^8}}{12k^5 + 3k^6} > 0$.

But, $\frac{\sqrt{9k^8}}{12k^5 + 3k^6} = \frac{3k^4}{15k^5} = \frac{1}{5k}$ and we know that $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges (use Integral Test or note that $p = 1$).

Therefore, the original series, which has larger terms, must diverge also.

(e) $\sum_{k=2}^{\infty} \frac{1}{k(\ln(k))^2}$  

[Integral Test]

$$\int_{2}^{\infty} \frac{1}{x(\ln x)^2} \, dx = \lim_{t \to \infty} \int_{2}^{t} \frac{1}{x(\ln x)^2} \, dx$$

Substitute $u = \ln x$, so $du = \frac{dx}{x}$.

$$= \lim_{t \to \infty} \int_{1}^{\ln t} \frac{1}{u^2} \, du$$

$$= \lim_{t \to \infty} \left[ -\frac{1}{u} \right]_{1}^{\ln t}$$

$$= \lim_{t \to \infty} \left[ -\frac{1}{\ln t} + \frac{1}{\ln 2} \right]$$

$$= 0 - \frac{1}{\ln 2}$$

$$= \frac{1}{\ln 2}$$

The integral converges, so the series must converge too.
Further, we know that \( \int_2^\infty \frac{1}{x(\ln x)^2} \, dx \leq \sum_{k=2}^\infty \frac{1}{k(\ln(k))^2} \leq a_2 + \int_2^\infty \frac{1}{x(\ln x)^2} \, dx. \)

Therefore, our lower bound is \( \int_2^\infty \frac{1}{x(\ln x)^2} \, dx = \frac{1}{\ln 2}. \)

And our upper bound is \( a_2 + \int_2^\infty \frac{1}{x(\ln x)^2} \, dx = \frac{1}{2(\ln 2)^2} + \frac{1}{\ln 2}. \)

7. Does the first series from the previous problem converge absolutely or conditionally?
\[
\sum_{k=1}^\infty \frac{|(-1)^k|}{\sqrt{k+1}} = \sum_{k=1}^\infty \frac{1}{\sqrt{k+1}},
\]
which diverges by the Integral Test (check for yourself).

Therefore, the first series from the previous problem converges conditionally.

8. Compute the radius and interval (including endpoints) of convergence for \( \sum_{k=1}^\infty \frac{(x+3)^k}{k \cdot 5^k} \).

\[
\lim_{k \to \infty} \frac{|(x+3)^{k+1}|}{(x+3)^k \cdot 5^{k+1}} = \lim_{k \to \infty} \left| \frac{x+3}{k+1} \cdot \frac{1}{5} \right| = \frac{|x+3|}{5}
\]

So, we are guaranteed convergence when \( \frac{|x+3|}{5} < 1. \) But this is equivalent to the following.
\[
-1 < \frac{x+3}{5} < 1
\]
\[-5 < x + 3 < 5
\]
\[-8 < x < 2
\]

To check convergence at the endpoints (where the Ratio Test is inconclusive), we plug in to the series itself.

At \( x = 2 \), we have \( \sum_{k=1}^\infty \frac{(2+3)^k}{k \cdot 5^k} = \sum_{k=1}^\infty \frac{5^k}{k \cdot 5^k} = \sum_{k=1}^\infty \frac{1}{k} \), which is the Harmonic Series and thus diverges.

At \( x = -8 \), we have \( \sum_{k=1}^\infty \frac{(-8+3)^k}{k \cdot 5^k} = \sum_{k=1}^\infty \frac{(-5)^k}{k \cdot 5^k} = \sum_{k=1}^\infty \frac{(-1)^k}{k} \), which converges by the Alternating Series Test.

Thus, the interval of convergence is \(-8 \leq x < 2\) and the radius of convergence is 5.

9. Find the complete Taylor series (in summation notation) for \( f(x) = \ln(1-x) \) about \( x = 0 \) and determine its interval of convergence.

\[
f(x) = \ln(1-x) \quad f(0) = 0
\]
\[
f'(x) = \frac{-1}{1-x} \quad f'(0) = -1
\]
\[
f''(x) = \frac{-1}{(1-x)^2} \quad f''(0) = -1
\]
\[
f'''(x) = \frac{-2}{(1-x)^3} \quad f'''(0) = -2
\]
\[
f^{(4)}(x) = \frac{-6}{(1-x)^4} \quad f^{(4)}(0) = -6
\]
Now plug in to the Taylor series formula with \( x_0 = 0 \).

\[
\begin{align*}
  f(x_0) + f'(x_0)x + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \ldots &= 0 - 1(x) + \frac{-1}{2}x^2 + \frac{-2}{6}x^3 + \\
  &= -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots \\
  &= \sum_{k=1}^{\infty} \frac{-x^k}{k}
\end{align*}
\]

We now find the interval of convergence as in the previous problem.

\[
\lim_{k \to \infty} \left| \frac{-\frac{x^{k+1}}{(k+1)!}}{-\frac{x^k}{k!}} \right| = \lim_{k \to \infty} \left| \frac{-x^{k+1}}{x^k} \right| = |x| < 1
\]

So, we are guaranteed convergence for \( |x| < 1 \), which is equivalent to \(-1 < x < 1\). Now check the endpoints.

At \( x = 1 \), we have \( \sum_{k=1}^{\infty} \frac{-1}{k} = -\sum_{k=1}^{\infty} \frac{1}{k} \), which is the negative of the Harmonic Series and thus diverges.

At \( x = -1 \), we have \( \sum_{k=1}^{\infty} \frac{(-1)^k}{k} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \), which converges by the Alternating Series Test.

So, the interval of convergence is \(-1 \leq x < 1\).

10. Write the complete series equal to \( \int_0^1 e^{-x^2} \, dx \) and show that it converges.

We know \( e^w = 1 + w + \frac{w^2}{2!} + \frac{w^3}{3!} + \ldots \), so by substitution we obtain the following.

\[
e^{-x^2} = 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \ldots = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \ldots
\]

Thus, \( \int_0^1 e^{-x^2} \, dx = \int_0^1 \left[ 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \ldots \right] \, dx \)

\[
= \left[ x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \ldots \right]_0^1 \\
= 1 - \frac{1^3}{3} + \frac{1^5}{5 \cdot 2!} - \frac{1^7}{7 \cdot 3!} + \ldots \\
= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)k!}
\]

The terms of this series alternate in sign.

And, \( 1 \geq \frac{1^3}{3} \geq \frac{1^5}{5 \cdot 2!} \geq \frac{1^7}{7 \cdot 3!} \geq \ldots \geq 0 \).

And, \( \lim_{k \to \infty} \frac{1}{(2k+1)k!} = 0 \).

Therefore, by the Alternating Series Test, the series must converge.
11. (Sections A and B may omit this question.) The probability density function (pdf) of the weights of newborn toads in a certain pond is given by \( f(x) = \frac{k}{(x+1)^4} \), where \( x \) is the weight (in ounces). Note that the domain is \( x \geq 0 \) since no toad can have a negative weight.

(a) What must be the value of \( k \)?

We know that the total area under any pdf must be 1 (because it must account for 100% of events.)

\[
\int_0^\infty \frac{k}{(x+1)^4} \, dx = \lim_{t \to \infty} \int_0^t \frac{k}{(x+1)^4} \, dx \\
= \lim_{t \to \infty} \left[ \frac{k(x+1)^{-3}}{-3} \right]_0^t \\
= \lim_{t \to \infty} \left( \frac{k}{-3(t+1)^3} - \frac{k}{-3(0+1)^3} \right) \\
= 0 - \frac{k}{-3} \\
= \frac{k}{3}
\]

So, we have \( k/3 = 1 \) or \( k = 3 \).

(b) What fraction of the newborn toads weigh more than one ounce?

\[
\int_1^\infty \frac{3}{(x+1)^4} \, dx = \lim_{t \to \infty} \int_1^t \frac{3}{(x+1)^4} \, dx \\
= \lim_{t \to \infty} \left[ \frac{3}{-1(t+1)^3} \right]_1^t \\
= \lim_{t \to \infty} \left( \frac{1}{-1(t+1)^3} - \frac{1}{-1(1+1)^3} \right) \\
= 0 - \frac{1}{-8} \\
= \frac{1}{8}
\]

Note that we could instead have computed \( 1 - \int_0^1 \frac{3}{(x+1)^4} \, dx \) and gotten the same answer.