1. (15 points) Let $W$ be the subspace spanned by the two vectors $\vec{u}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ and $\vec{u}_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$.

Let $\vec{y} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$.

(a) Is the vector $\vec{y}$ in $W$? Explain.

(b) Find a vector in $W$ that is closest to $\vec{y}$.

(c) Find a vector that is orthogonal to $W$. 
2. (15 points) Orthogonally diagonalize the matrix $A = \begin{bmatrix} -7 & 24 \\ 24 & 7 \end{bmatrix}$. (The eigenvalues of $A$ are 25 and $-25$.)
3. (12 points) Determine if the following sets are subspaces of the appropriate vector spaces. If a set is a subspace, find a basis and the dimension of the subspace.

(a) \[ W = \left\{ \begin{bmatrix} 2a - c + d \\ b - 2c - 2d \\ a + 3b + d \\ 2b + c + d \end{bmatrix} : a, b, c, d \text{ are real numbers} \right\}. \]

(b) All polynomials in \( \mathbb{P}_3 \) of the form \( t + a \).
4. (12 points) Let $\vec{p}_1(t) = 2t - t^2$, $\vec{p}_2(t) = 2t$, $\vec{p}_3(t) = 2 - t$.

(a) Use coordinate vectors to show that $\mathcal{B} = \{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$ is a basis for $\mathbb{P}_2$.

(b) Find the polynomial $\vec{q}$ in $\mathbb{P}_2$, given that $[q]_B = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$. 
5. (10 points) Suppose \( \{ \vec{v}_1, \vec{v}_2 \} \) is a linearly independent set in \( \mathbb{R}^7 \).

(a) Show that \( \{ \vec{v}_1 + \vec{v}_2, \vec{v}_1 - \vec{v}_2 \} \) is also a linearly independent set.

(b) Is \( \vec{v}_1 \) in \( \text{Span}\{ \vec{v}_1 + \vec{v}_2, \vec{v}_1 - \vec{v}_2 \} \)? Explain.

6. (8 points) Suppose \( U \) is an \( n \times n \) orthogonal matrix. For every vector \( \vec{x} \) in \( \mathbb{R}^n \), show that the length of the vector \( U\vec{x} \) is the same as the length of the vector \( \vec{x} \). (Hint: Length of \( U\vec{x} \) is \( \sqrt{U\vec{x} \cdot U\vec{x}} \) and length of \( \vec{x} \) is \( \sqrt{\vec{x} \cdot \vec{x}} \). So it is enough to show that \( U\vec{x} \cdot U\vec{x} = \vec{x} \cdot \vec{x} \).)
7. (10 points) Let \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be a linear transformation given by \( T(x) = Ax \) (\( A \) is a 2 \( \times \) 2 matrix). Figure I below shows vectors \( \vec{u}, \vec{v} \) and \( \vec{w} \) and Figure II below shows vectors \( T(\vec{u}), T(\vec{v}), \) and \( T(\vec{w}) \). Use this information to answer the questions that follow.

(a) In Figure II, draw \( T(\vec{v} + \vec{w}) \).

(b) Which of the vectors \( \vec{u}, \vec{v} \) and \( \vec{w} \) (if any) are eigenvectors of \( A \)? What are the corresponding eigenvalues? Explain.

(c) Is \( T \) one-to-one? Explain.
8. (18 points) Short answers: (No explanations needed. Simply write your answers. If you do some calculation to get the answer, show the calculation.)

(a) Find the distance between the vector \( \vec{u} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \) and the vector \( \vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \).

(b) A \( 2 \times 4 \) matrix has rank 2. Find the dimension of the null space of this matrix.

(c) Suppose \( A \) is a \( 4 \times 4 \) matrix with \( \det A = 20 \). What is \( \det 3A \)?

(d) Let \( B \) be a \( 5 \times 5 \) matrix. The dimension of the eigenspace corresponding to the eigenvalue \(-3\) of \( B \) is 2. What is the dimension of \( \text{Nul} \ (B + 3I) \)? (Here \( I \) is the \( 5 \times 5 \) identity matrix.)

(e) In the following figure, draw the orthogonal projection of the vector \( \vec{u} \) onto the subspace spanned by the vector \( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \).
(f) Let $T : \mathbb{R}^4 \to \mathbb{R}^2$ be the linear transformation given by

$$T(x_1, x_2, x_3, x_4) = (x_1 + x_2 - 2x_4, 2x_3 + x_4).$$

What is the standard matrix of $T$?

(g) Let $T : M_{2\times2} \to \mathbb{R}^2$ be the linear transformation defined by

$$T \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} a - c \\ b - d \end{bmatrix}.$$ 

i. Find $T \left( \begin{bmatrix} -2 & 5 \\ 0 & 20 \end{bmatrix} \right)$.

ii. Let $\vec{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Find a matrix $A$ in $M_{2\times2}$ such that $T(A) = \vec{b}$. 