Your Name:

There are 9 problems in this exam. **Please complete only eight**, clearly crossing out the one you do not wish to be graded. If you do not clearly indicate, your highest-scoring problem will be omitted! (Follow directions and avoid the grader’s wrath.) Extra credit will **not** be awarded for attempting a ninth problem.

On each problem, you must show all your work, or otherwise thoroughly explain your conclusions. **Except where noted, include at least one complete English sentence in every solution.** Units may be requested for your final answer; a point deduction will apply if they are omitted or incorrect.

You may use a calculator on any portion of this exam. However, this does not excuse you from showing your work at any time. **Your calculating skills are at issue on this exam, not your calculator’s.**

You will have two hours to complete this exam.

<table>
<thead>
<tr>
<th>Question</th>
<th>Possible</th>
<th>Your Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>25</td>
<td></td>
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<tr>
<td>2</td>
<td>25</td>
<td></td>
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<td>3</td>
<td>25</td>
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<td>4</td>
<td>25</td>
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<td>25</td>
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<td>25</td>
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<td>25</td>
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<td>9</td>
<td>25</td>
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<td><strong>Total</strong></td>
<td><strong>200</strong></td>
<td>**</td>
</tr>
</tbody>
</table>
Problem 1. (25 points) Rabbits are commonly thought to multiply more quickly than your average animal. Suppose a colony of rabbits has total population \( P(t) \), measured in thousands of rabbits, after \( t \) months, according to
\[
P(t) = \frac{1}{102 - 2t}.
\]

(a) (5 points) What is the domain of this function? What does it mean for the rabbits?

The “domain” of a function is the set of all input values for which it can be defined. Here, \( P(t) \) is defined for all \( t \) unless the denominator of the fraction becomes zero, which happens when
\[
102 - 2t = 0 \quad \text{or} \quad t = 51.
\]

The domain of this function is \( \{ t \in \mathbb{R} : t \neq 51 \} \). This model, then, cannot predict the population of rabbits at \( t = 51 \) months.

(You might also argue that it cannot predict the rabbit population for \( t < 0 \), since this is previous to the experiment, and it cannot predict what happens for \( t > 51 \) since \( P(t) \) is negative for those values of \( t \) — and you can’t have negative rabbits.)

(b) (10 points) What happens to the rabbit population near the point(s) where \( P(t) \) is undefined? How can you tell?

As \( t \to 51^- \), the population of rabbits blows up to infinity:
\[
\lim_{t \to 51^-} \frac{1}{102 - 2t} \sim \frac{1}{0^+} = +\infty.
\]

(This is called a “population catastrophe.”) Again, we can tell because the closer \( t \) gets to 51 (from the left), the smaller positive number appears in the denominator, making the fraction increase without bound.

(c) (10 points) Verify that the rabbits’ population satisfies the differential equation \( P' = 2P^2 \).

Computing both sides of this equation, we have
\[
\begin{align*}
P'(t) &= \frac{-1}{(102 - 2t)^2}(-2) \\
&= \frac{2}{(102 - 2t)^2}
\end{align*}
\]
\[
\begin{align*}
2P(t)^2 &= 2\left(\frac{1}{102 - 2t}\right)^2 \\
&= \frac{2}{(102 - 2t)^2}
\end{align*}
\]

Since both sides are the same, this verifies that \( P \) satisfies the given differential equation.

(Note: the typical model of population growth is \( P' = kP \); increasing the power of \( P \) on the right-hand side results in much more rapid population growth. In fact, putting any value of \( r > 1 \) into \( P' = kP^r \) results in the kind of population catastrophe seen in this problem.)
Problem 2. (25 points) Shown below is a graph of the derivative of a function \( T \). Answer each of the following questions; on this page only, you do not need to provide explanations except where indicated.

(a) (3 points) \( T \) is decreasing on what interval(s)?

\((-7, -1)\)

(b) (3 points) \( T \) is concave up on what interval(s)?

\((6, 9)\)

(c) (3 points) \( T \) has a local minimum at what point(s)?

\( t = -1 \)

(Here, \( T \) switches from decreasing to increasing — even though \( T' \) doesn’t cross through zero!)

(d) (3 points) \( T \) has an inflection at what point(s)?

\( t = 6 \)

(e) (3 points) How can you tell at which point(s) \( T(x) = 0 \)? Explain.

In short, we cannot without more information! If all we know is the derivative \( T' \), this only tells us the shape of the curve \( T(x) \) and not its location. \( T \) is just an antiderivative of \( T' \), so if we do not know which antiderivative it is, we cannot answer this question.

(f) (10 points) On the axes provided, sketch a possible graph of the function \( T \). Do not worry about the scale on the vertical axis, only the major features of the graph’s shape.
Problem 3. (25 points) For the function $g(x) = \sqrt{x}$:

(a) (10 points) Complete the table of data below, and use average rates of change to speculate on the value of the derivative $g'(0)$. (Do not compute $g'(x)$ directly!)

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>0.001</th>
<th>0.01</th>
<th>0.1</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g(x)$</td>
<td>0</td>
<td>0.0316</td>
<td>0.1</td>
<td>0.316</td>
<td>1</td>
</tr>
</tbody>
</table>

Let’s look at the average rates of change on, say, the intervals $[0, 0.1]$, $[0, 0.01]$, and $[0, 0.001]$:

\[
\frac{g(0.1) - g(0)}{0.1 - 0} = \frac{0.316}{0.1} = 3.16
\]

\[
\frac{g(0.01) - g(0)}{0.01 - 0} = \frac{0.1}{0.01} = 10
\]

\[
\frac{g(0.001) - g(0)}{0.001 - 0} = \frac{0.0316}{0.001} = 31.6
\]

You could make several predictions here, but the most tempting is to say that these average rates of change appear to be getting bigger and bigger. We suspect that the closer we get to $x = 0$, the larger the average rate of change gets. We speculate that $g'(0)$ becomes infinite.

(b) (15 points) Using the limit definition of derivative, determine $g'(0)$. Explain how this connects with your answer from part (a).

If $g'(0)$ exists, it is equal to the limit

\[
\lim_{h \to 0} \frac{g(0 + h) - g(0)}{h} = \lim_{h \to 0} \frac{\sqrt{0 + h} - \sqrt{0}}{h}
\]

\[
= \lim_{h \to 0} \frac{\sqrt{h}}{h}
\]

\[
= \lim_{h \to 0} \frac{1}{\sqrt{h}}
\]

\[
\sim \frac{1}{0^+} \to +\infty.
\]

This confirms that the closer $h$ gets to zero, the larger the average rate of change on the interval $[0, h]$ gets, without bound. The derivative $g'(0)$ blows up to infinity (so it does not exist as a number).
Problem 4. (25 points) Compute the derivative $\frac{dy}{dx}$ for each of the following relationships between $x$ and $y$. You do not need to algebraically simplify your answers.

(a) (6 points) $y = 8e^x + e^{8x} - 2x^3 + \frac{3}{x^2} + \ln x$

$$y' = 8e^x + 8e^{8x} - 6x^2 - \frac{6}{x^3} + \frac{1}{x}$$

(b) (6 points) $y = \arctan(\sqrt{x^2 + 16})$

$$y' = \frac{x}{(17 + x^2) \sqrt{x^2 + 16}}$$

(c) (6 points) $y = \frac{\cos x}{e^{2x} - e^{-2x}}$

$$y' = \frac{-\sin x(e^{2x} - e^{-2x}) - \cos x(2e^{2x} + 2e^{-2x})}{(e^{2x} - e^{-2x})^2}$$

(d) (7 points) $y = x^{17x} \sin(17x)$

$$y' = 17x^{16}17^x \sin(17x) + x^{17}17^x(\ln 17) \sin(17x) + x^{17}17^x(17 \cos(17x))$$
Problem 5. (25 points)

(a) (10 points) The curve graphed below has equation \(\sin(xy) + x^2 + y^2 = 2\). Find an equation for a line tangent to this curve at the point \((x, y) = (0.827, 0.827)\).

We only need two things to determine an equation for a line: its slope and any point it passes through. The point is easy: \((x, y) = (0.827, 0.827)\). The slope is given by the derivative \(dy/dx\).

Using implicit differentiation with respect to \(x\),

\[
\cos(xy)(y + x \frac{dy}{dx}) + 2x + 2y \frac{dy}{dx} = 0
\]

\[
\frac{dy}{dx}(x \cos(xy) + 2y) = -y \cos(xy) - 2x
\]

\[
\frac{dy}{dx} = -\frac{y \cos(xy) + 2x}{x \cos(xy) + 2y}
\]

Evaluating at \(x = 0.827\) and \(y = 0.827\) gives

\[
\frac{dy}{dx} = -\frac{0.827 \cos(0.827^2) + 2(0.827)}{0.827 \cos(0.827^2) + 2(0.827)} = -1.
\]

Thus the tangent line has slope \(-1\) and passes through \((0.827, 0.827)\), so its equation is

\[
y - 0.827 = -1(x - 0.827) \quad \text{or} \quad y = 1.654 - x.
\]

(b) (15 points) As \(x \to 0\), both of the functions \(f(x) = e^{x^2} - 1\) and \(g(x) = \cos(x^2) - 1\) approach zero. Precisely determine which gets to zero more quickly, by computing an appropriate limit.

To judge this footrace, we compute the limit \(\lim_{x \to 0} \frac{f(x)}{g(x)}\). If this limit is zero, \(f\) has won the race. If it is infinite, \(g\) has won the race. If it is neither, the functions have “tied.”

Recall that l’Hôpital’s rule assists us whenever a limit has the indeterminate form \(\frac{0}{0}\). Each use of l’Hôpital’s rule is marked with a \(\frac{0}{0}\) below.

\[
\lim_{x \to 0} \frac{e^{x^2} - 1}{\cos(x^2) - 1} \xrightarrow{\frac{0}{0}} \lim_{x \to 0} \frac{2xe^{x^2}}{-2x \sin(x^2)}
\]

\[
= \lim_{x \to 0} \frac{e^{x^2}}{-\sin(x^2)}
\]

\[
\sim \frac{1}{0^-} = -\infty
\]

Since the limit is infinite, \(g(x) = \cos(x^2) - 1\) wins the race: it approaches zero strictly faster than \(f(x)\) does.
Problem 6. (25 points) After this exam is over, you plan to jump in a chartered Learjet-31 to fly home for the holidays. It’s currently parked on the tarmac at Lewiston/Auburn’s airport and waiting to take you to Las Vegas, a flight of 2400 miles.

Unfortunately, both the jet and the fuel it burns are expensive. The Learjet costs $1500 an hour to charter. Furthermore, if it flies at an airspeed of \( x \) mph, it consumes \( (100 + \frac{x^2}{900}) \) gallons of jet fuel per hour — and jet fuel costs $5 per gallon.

What is the smallest total cost for this trip? Hint: how does the total time of the trip relate to the airspeed \( x \)?

**Step 1: State your objective.** We wish to find the minimum of the total cost function. We will use the airspeed \( x \) as the independent variable.

**Step 2: Visualize success.** Here, we’ll use a quick diagram to organize the two parts of the total cost function. If \( h \) is the number of hours the flight will take, the total cost is made up of...

<table>
<thead>
<tr>
<th>cost of charter</th>
<th>and</th>
<th>cost of fuel</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1500 per hour for ( h ) hours, a total of ( 1500h )</td>
<td>((100 + \frac{x^2}{900})) gallons of fuel per hour, at $5 each, for ( h ) hours, a total of ( 5(100 + \frac{x^2}{900})h )</td>
<td></td>
</tr>
</tbody>
</table>

Therefore the total cost function can be written as

\[
C = 1500h + 5(100 + \frac{x^2}{900})h = \left(2000 + \frac{x^2}{180}\right)h.
\]

**Step 3: Eliminate the competition.** We have one too many variables: we need to either eliminate \( h \) or eliminate \( x \). Either choice will work: for fun, let’s eliminate \( h \). What relationship exists between the total time of the flight \( h \) and the airspeed \( x \)? The total distance, 2400 miles, is the product of the two:

\[
2400 = hx \quad \text{so} \quad h = \frac{2400}{x}.
\]

Substituting this into the total cost function gives us the single-variable function

\[
C(x) = \left(2000 + \frac{x^2}{180}\right) \frac{2400}{x} = \frac{4800000}{x} + \frac{40}{3}x.
\]

**Step 4: Optimize.**

The stationary points of \( C \) will occur when \( C'(x) = 0 \). Where is this?

\[
C'(x) = -\frac{4800000}{x^2} + \frac{40}{3} = 0
\]

\[
\frac{4800000}{x^2} = \frac{40}{3}
\]

\[
x^2 = 360000
\]

\[
x = \pm \sqrt{360000} = \pm 600.
\]

Since \( x \), our airspeed, ought to be positive, we choose \( x = 600 \) mph.

Is this a local minimum? A quick check of the second derivative shows it is, since \( C \) is concave up for all positive values of \( x \):

\[
C''(x) = \frac{9600000}{x^3} \quad \text{so} \quad C''(600) = \frac{2}{45} > 0.
\]

Thus the cost is minimized at an airspeed of \( x = 600 \) mph. At this airspeed, the cost of the flight is

\[
C(600) = \frac{4800000}{600} + \frac{40}{3}(600) = 16000
\]

The cheapest possible charter to Vegas, then, will cost $16,000.
Problem 7. (25 points) Water, it is said, rolls off a duck’s back. If the duck is shaped like the curve \( x^2 + 4y^2 = 169 \) shown below, and a drop of water is at the point \((x, y) = (5, 6)\), and the drop has a downward velocity of 1.2 cm/sec, what is its horizontal velocity at that moment in time?

Step 1: What quantities are being related? The position of this water drop has an \( x \) component and a \( y \) component. As it rolls off the duck, both \( x \) and \( y \) depend on time, \( t \).

Step 2: Know what you know, know what you don’t know. We are given the position of the water drop: \( x = 5 \), \( y = 6 \). We are also given its downward velocity: this is the rate of change of \( y \) with respect to time. Namely,

\[
\frac{dy}{dt} = -1.2.
\]

(Negative sign indicates the drop is falling, not rising!) What we don’t know is the horizontal velocity \( \frac{dx}{dt} \).

We organize this information in the following table:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5 cm</td>
<td>6 cm</td>
</tr>
<tr>
<td>( \frac{dx}{dt} )</td>
<td>( \frac{dy}{dt} )</td>
</tr>
<tr>
<td>?</td>
<td>-1.2 cm/sec</td>
</tr>
</tbody>
</table>

Step 3: Relate the quantities. This is already done for us by the equation of the curve given:

\[
x^2 + 4y^2 = 169
\]

Step 4: Implicit differentiation. Using implicit differentiation with respect to time \( t \), we obtain the all-important rate equation:

\[
2x \frac{dx}{dt} + 8y \frac{dy}{dt} = 0
\]

Now substituting the three known quantities, we can solve for \( \frac{dx}{dt} \):

\[
2(5) \frac{dx}{dt} + 8(6)(-1.2) = 0
\]

\[
10 \frac{dx}{dt} - 57.6 = 0
\]

\[
\frac{dx}{dt} = \frac{57.6}{10} = 5.76 \text{ cm/sec.}
\]
Problem 8. (25 points) Compute the following antiderivatives and definite integrals.

(a) (9 points) All antiderivatives of  \( \frac{4}{x^2} + \frac{2}{x} - \sin x \)

\[ -\frac{4}{x} + 2 \ln |x| + \cos x + C \]

(b) (8 points) The area function \( \int_{0}^{x} t^2 + 2^t \, dt \)

\[ \frac{x^3}{3} + \frac{2^x}{\ln 2} - 1 \]

(c) (8 points) The definite integral \( \int_{-1}^{4} 18x^8 - 2 \, dx \)

\[ 2x^9 - 2x \bigg|_{-1}^{4} = 524280 - 0 = 524280 \]
Problem 9. (25 points) While you have been taking this exam, a tiny device implanted in the ceiling has been measuring your happiness every 10 minutes. The results of its measurements are shown in the table below.

<table>
<thead>
<tr>
<th>minutes since exam began</th>
<th>0</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>60</th>
<th>70</th>
<th>80</th>
<th>90</th>
</tr>
</thead>
<tbody>
<tr>
<td>h(t)</td>
<td>9.5</td>
<td>6.4</td>
<td>4.4</td>
<td>3.5</td>
<td>2.9</td>
<td>2.7</td>
<td>2.6</td>
<td>3.1</td>
<td>4.6</td>
<td>6.9</td>
</tr>
</tbody>
</table>

(a) (15 points) Use the trapezoidal rule with 9 subintervals to estimate the integral \( \int_0^{90} h(t) \, dt \).

The trapezoidal approximation is the average of left- and right-hand approximations. We’ll compute each. With 9 subintervals on the interval \([0, 90]\), each must have a width of \( \Delta t = \frac{90-0}{9} = 10 \). Luckily for us, the data in the table are already spaced apart by this step size.

The left- and right-hand sums are then

\[
L_9 = \sum_{n=0}^{8} h(10n) \frac{10}{10} = 9.5(10) + 6.4(10) + 4.4(10) + \cdots + 3.1(10) + 4.6(10) + 6.9(10) = 397
\]

\[
R_9 = \sum_{n=1}^{9} h(10n) \frac{10}{10} = 6.4(10) + 4.4(10) + 3.5(10) + \cdots + 2.9(10) + 2.7(10) + 2.6(10) + \cdots + 3.1(10) + 4.6(10) + 6.9(10) = 371
\]

So the trapezoidal approximation is the average

\[
T_9 = \frac{L_9 + R_9}{2} = \frac{397 + 371}{2} = 384.
\]

(b) (5 points) What are the units of your answer to part (a), and how can this integral be interpreted?

The units of an integral — being an area on the graph of a function — are the product of the units of input and units of output. So here, the units are “happiness-minutes”. We might interpret the answer “384 happiness minutes” as being the total amount of accumulated happiness you experienced while taking this exam.

(c) (5 points) If you are convinced that \( h(t) \) is a concave up function, what can you say about the approximation in part (a)?

The trapezoidal approximation will always overestimate the integral of a concave up function, since concave up functions lie beneath their secant lines. We conclude that your actual total accumulated happiness during the exam is probably less than 384 happiness-minutes.