1. Use a second-degree Taylor polynomial to estimate $\sqrt[3]{28}$.

We choose $f(x) = \sqrt[3]{x}$ and $x_0 = 27$ because 27 is the perfect cube closest to 28.

$$f(x) = x^{1/3}$$
$$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$$
$$f'(27) = \frac{1}{3 \cdot 27^{2/3}} = \frac{1}{27}$$
$$f''(x) = -\frac{2}{9}x^{-5/3} = -\frac{2}{9x^{5/3}}$$
$$f''(27) = -\frac{2}{9 \cdot 27^{5/3}} = -\frac{2}{2187}$$

Now plug in to the Taylor polynomial formula with $x_0 = 27$.

$$P_2(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 = 3 + \frac{1}{27}(x - 27) - \frac{1}{2187}(x - 27)^2$$

Finally, evaluate at $x = 28$.

$$\sqrt[3]{28} \approx P_2(28) = 3 + \frac{1}{27}(28 - 27) - \frac{1}{2187}(28 - 27)^2 = \frac{6641}{2187} \approx 3.0365797$$

2. What is the largest possible error that could have occurred in your previous estimate?

We know that $|f(x) - P_n(x)| \leq \frac{K_{n+1}}{(n+1)!}|x - x_0|^{n+1}$.

In this case, $n = 2$, $x_0 = 27$, and $x = 28$.

$$K_3 = \text{max of } |f'''(x)| \text{ on } [27, 28] = \text{max of } \left| \frac{10}{27x^{8/3}} \right| \text{ on } [27, 28] = \frac{10}{27 \cdot 27^{8/3}} = \frac{10}{177147}$$

Putting this all together, we have $|f(x) - P_2(x)| \leq \frac{10}{3!} |28 - 27|^3 = \frac{5}{531441} \approx 0.0000094$.

3. Use a comparison to show whether each of the following converges or diverges. If an integral converges, give a good upper bound for its value.

(a) $\int_1^\infty \frac{7 + 5\sin x}{x^2} \, dx$

For all $x \geq 1$, we have $0 \leq \frac{7 + 5\sin x}{x^2} \leq \frac{7 + 5(1)}{x^2} = \frac{12}{x^2}$ because the maximum of $\sin x$ is 1.

$$12 \int_1^\infty \frac{dx}{x^2} = 12 \lim_{t \to \infty} \int_1^t \frac{dx}{x^2}$$
$$= 12 \lim_{t \to \infty} \left. -\frac{1}{x} \right|_1^t$$
$$= 12 \lim_{t \to \infty} \left[ -\frac{1}{t} - \frac{1}{1} \right]$$
$$= 12[0 - (-1)]$$
$$= 12$$

Therefore, the original integral in question must converge to a value less than 12.

(b) $\int_1^\infty \frac{1 + 3x^2 + 2x^3}{\sqrt{10x^{12} + 17x^{10}}} \, dx$

For $x \geq 1$, we have $\frac{1 + 3x^2 + 2x^3}{\sqrt{10x^{12} + 17x^{10}}} \geq \frac{2x^3}{\sqrt{10x^{12} + 17x^{10}}} \geq 0$. (We've made the numerator smaller and the denominator larger, so the new fraction is smaller.)
But \( \frac{2x^3}{\sqrt{10x^4 + 17x^2}} = \frac{2x^3}{\sqrt{27x^2}} = \frac{2}{3x} \) and we know that \( \frac{2}{3} \int_1^\infty \frac{dx}{x} \) diverges (compute for yourself or notice that \( p = 1 \)).

Therefore the original integral must also diverge.

4. The probability density function (pdf) of the weights of newborn toads in a certain pond is given by \( f(x) = \frac{k}{(x+1)^4} \), where \( x \) is the weight (in ounces). Note that the domain is \( x \geq 0 \) since no toad can have a negative weight.

(a) **What must be the value of \( k \)?**

We know that the total area under any pdf must be 1 (because it must account for 100\% of events.)

\[
\int_0^\infty \frac{k}{(x+1)^4} \, dx = \lim_{t \to \infty} \int_0^t \frac{k}{(x+1)^4} \, dx
\]

\[
= \lim_{t \to \infty} k \frac{(x+1)^{-3}}{-3} \bigg|_0^t
\]

\[
= \lim_{t \to \infty} \frac{k}{-3(x+1)^3} \bigg|_0^t
\]

\[
= \lim_{t \to \infty} k - 3(t+1)^3 - k - 3(0+1)^3
\]

\[
= 0 - \frac{k}{3}
\]

\[
= \frac{k}{3}
\]

So, we have \( k/3 = 1 \) or \( k = 3 \).

(b) **What fraction of the newborn toads weigh more than one ounce?**

\[
\int_1^\infty \frac{3}{(x+1)^4} \, dx = \lim_{t \to \infty} \int_1^t \frac{3}{(x+1)^4} \, dx
\]

\[
= \lim_{t \to \infty} \frac{1}{-3(x+1)^3} \bigg|_1^t
\]

\[
= \lim_{t \to \infty} \frac{1}{-1(t+1)^3} - \frac{1}{-1(1+1)^3}
\]

\[
= 0 - \frac{1}{-8}
\]

\[
= \frac{1}{8}
\]

Note that we could instead have computed \( 1 - \int_0^1 \frac{3}{(x+1)^4} \, dx \) and gotten the same answer.

5. Decide if each of the following sequences \( \{a_k\} \) converges or diverges. If a sequence converges, compute its limit.

(a) \( a_k = 3 + \frac{1}{10^k} \)

Terms are 3.1, 3.01, 3.001, 3.0001, ...  

\[
\lim_{k \to \infty} \left( 3 + \frac{1}{10^k} \right) = 3,
\]

so the sequence converges to 3.

(b) \( a_k = (-1)^k \)

Terms are \(-1, 1, -1, 1, ... \)

\[
\lim_{k \to \infty} (-1)^k
\]

doesn’t exist, so the sequence diverges.
6. Decide if each of the following series converges or diverges. If a series converges, find its value.

(a) \(3 + 3.01 + 3.001 + 3.0001 + \ldots\)
\[\lim_{k \to \infty} a_k = 3 \neq 0\], so the series diverges by the nth Term Test. (We keep adding 3's forever.)

[Compare this with the first sequence of the previous problem.]

(b) \(1 + 1/2 + 1/3 + 1/4 + \ldots\)

This is the famous Harmonic Series, which diverges even though the terms do approach 0. We can use the Integral Test:
\[\int_1^{\infty} \frac{1}{x} \, dx\]
diverges, which means that \(\sum_{k=1}^{\infty} \frac{1}{k}\) must diverge too.

(c) \(5 − 5/3 + 5/9 − 5/27 + \ldots\)

This is a geometric series with \(r = -\frac{1}{3}\), so it converges to \(\frac{a}{1-r} = \frac{5}{1-(-1/3)} = \frac{15}{4}\).

7. Decide if each of the following series converges or diverges. If a series converges, find upper and lower bounds for its value.

(a) \(\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k+1}}\) [Alternating Series Test]

The terms of this series alternate in sign.
And, \(\frac{1}{\sqrt{2}} \geq \frac{1}{\sqrt{3}} \geq \frac{1}{\sqrt{4}} \geq \ldots \geq 0\).
And, \(\lim_{k \to \infty} \frac{1}{\sqrt{k+1}} = 0\).

Therefore, by the Alternating Series Test, the series must converge.
We know that any two consecutive partial sums will provide upper and lower bounds:
lower bound = \(S_1 = -\frac{1}{\sqrt{2}}\)  upper bound = \(S_2 = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}}\)
[To get better bounds, use later partial sums, such as \(S_5\) and \(S_6\).]

(b) \(\sum_{k=1}^{\infty} \frac{(2k)!}{3^k(k!)^2}\) [Ratio Test]

\[\lim_{k \to \infty} \frac{a_{k+1}}{a_k} = \frac{[2(k+1)]!3^k(k!)^2}{3^{k+1}[(k+1)!]^2(2k)!} \]
\[= \lim_{k \to \infty} \frac{(2k + 2)!}{(2k)!} \frac{3^k}{3^{k+1}} \frac{(k!)^2}{[(k+1)!]^2} \]
\[= \lim_{k \to \infty} \frac{(2k + 2)(2k + 1)}{1} \frac{1}{3} \frac{1}{(k + 1)^2} \]
\[= \frac{4k^2 + 6k + 2}{3k^2 + 6k + 3} \]
Use L'Hopital or divide each term by \(k^2\).
\[= \frac{4}{3}\]
Since the limit of the ratio is greater than 1, the series diverges.
(c) \( \sum_{k=1}^{\infty} \left( \frac{1}{100} + \frac{1}{k^5} \right) \)

\[ \lim_{k \to \infty} \left( \frac{1}{100} + \frac{1}{k^5} \right) = \frac{1}{100} \neq 0 \], so, by the nth Term Test, the series diverges.

(d) \( \sum_{k=1}^{\infty} \frac{\sqrt{9k^5 + 5k^3}}{12k^5 + 3} \)

For \( k \geq 1 \), we have \( \frac{\sqrt{9k^5 + 5k^3}}{12k^5 + 3} > \frac{\sqrt{9k^5}}{12k^5 + 3k^3} > 0 \).

But, \( \frac{\sqrt{9k^5}}{12k^5 + 3k^3} = \frac{3k^4}{15k^3} = \frac{1}{5} \) and we know that \( \sum_{k=1}^{\infty} \frac{1}{k} \) diverges (use Integral Test or note that \( p = 1 \)).

Therefore, the original series, which has larger terms, must diverge also.

(e) \( \sum_{k=2}^{\infty} \frac{1}{k(ln(k))^2} \)

\[ \int_2^{\infty} \frac{1}{x(ln(x))^2} \, dx = \lim_{t \to \infty} \int_2^{t} \frac{1}{x(ln(x))^2} \, dx \]

\[ = \lim_{t \to \infty} \int_{x=2}^{x=t} \frac{1}{u^2} \, du \quad \text{Substitute } u = \ln x, \text{ so } du = \frac{dx}{x}. \]

\[ = \lim_{t \to \infty} \left[ \frac{-1}{\ln u} \right]_{x=2}^{x=t} \]

\[ = \lim_{t \to \infty} \left[ -\frac{1}{\ln t} - \frac{1}{\ln 2} \right] \]

\[ = 0 - \frac{1}{\ln 2} \]

\[ = \frac{1}{\ln 2} \]

The integral converges, so the series must converge too.

Further, we know that \( \int_2^{\infty} \frac{1}{x(ln(x))^2} \, dx \leq \sum_{k=2}^{\infty} \frac{1}{k(ln(k))^2} \leq a_2 + \int_2^{\infty} \frac{1}{x(ln(x))^2} \, dx. \)

Therefore, our lower bound is \( \int_2^{\infty} \frac{1}{x(ln(x))^2} \, dx = \frac{1}{\ln 2}. \)

And our upper bound is \( a_2 + \int_2^{\infty} \frac{1}{x(ln(x))^2} \, dx = \frac{1}{2(ln(2))^2} + \frac{1}{\ln 2}. \)

8. Does the first series from the previous problem converge absolutely or conditionally?

Recall that a series \( \sum_{k=1}^{\infty} a_k \) converges \textit{absolutely} if \( \sum_{k=1}^{\infty} |a_k| \) converges; a series \( \sum_{k=1}^{\infty} a_k \) converges \textit{conditionally} if \( \sum_{k=1}^{\infty} a_k \) converges but \( \sum_{k=1}^{\infty} |a_k| \) diverges.

\[ \sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k} + 1} \]

which diverges by the Integral Test (check for yourself).

Therefore, the first series from the previous problem converges \textit{conditionally}. 
9. Compute the radius and interval (including endpoints) of convergence for \( \sum_{k=1}^{\infty} \frac{(x+3)^k}{k \cdot 5^k} \).

\[
\lim_{k \to \infty} \left| \frac{(x+3)^{k+1}}{(k+1) \cdot 5^{k+1}} \cdot \frac{k}{5^k} \right| = \lim_{k \to \infty} \left| \frac{(x+3)^{k+1}}{(x+3)^k} \cdot \frac{k}{k+1} \cdot \frac{5^k}{5^{k+1}} \right| = \left| \frac{x+3}{5} \right|
\]

Use L’Hopital on the middle fraction.

\[
= \left| \frac{x+3}{5} \right| = \frac{|x+3|}{5}
\]

So, we are guaranteed convergence when \( \frac{|x+3|}{5} < 1 \). But this is equivalent to the following.

\[-1 < \frac{x+3}{5} < 1 \]
\[-5 < x+3 < 5 \]
\[-8 < x < 2 \]

To check convergence at the endpoints (where the Ratio Test is inconclusive), we plug in to the series itself.

At \( x = 2 \), we have \( \sum_{k=1}^{\infty} \frac{(2+3)^k}{k \cdot 5^k} = \sum_{k=1}^{\infty} \frac{5^k}{k \cdot 5^k} = \sum_{k=1}^{\infty} \frac{1}{k} \), which is the Harmonic Series and thus diverges.

At \( x = -8 \), we have \( \sum_{k=1}^{\infty} \frac{(-8+3)^k}{k \cdot 5^k} = \sum_{k=1}^{\infty} \frac{(-5)^k}{k \cdot 5^k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \), which converges by the Alternating Series Test.

Thus, the interval of convergence is \( -8 \leq x < 2 \) and the radius of convergence is 5.

10. Find the complete Taylor series (in summation notation) for \( f(x) = \ln(1-x) \) about \( x = 0 \).

\[
f(x) = \ln(1-x) \quad f(0) = 0
\]
\[
f'(x) = \frac{-1}{1-x} \quad f'(0) = -1
\]
\[
f''(x) = \frac{-1}{(1-x)^2} \quad f''(0) = -1
\]
\[
f'''(x) = \frac{-2}{(1-x)^3} \quad f'''(0) = -2
\]
\[
f^{(4)}(x) = \frac{-6}{(1-x)^4} \quad f^{(4)}(0) = -6
\]

Now plug in to the Taylor series formula with \( x_0 = 0 \).

\[
f(x) = f(x_0) + f'(x_0) x + \frac{f''(x_0)}{2!} (x-x_0)^2 + \frac{f'''(x_0)}{3!} (x-x_0)^3 + \ldots = 0 - 1(x) + \frac{-1}{2}x^2 + \frac{-2}{6}x^3 + \ldots
\]
\[
= -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots
\]
\[
= \sum_{k=1}^{\infty} \frac{-x^k}{k}
\]

11. Evaluate the following exactly.
(a) \[ 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \ldots \]

We recall that \(e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots.\)

The series in question is the series for \(e^x\) with \(x = -1\); therefore, it converges to \(e^{-1}\).

(b) \[ \frac{8}{3} - \frac{8}{9} + \frac{8}{27} - \frac{8}{81} + \ldots \]

We recognize this as a geometric series with \(r = -1/3\) and \(a = 8/3\); therefore, it converges to \(\frac{a}{1-r} = \frac{8/3}{1 - (-1/3)} = 2\).

(c) \[ 1 - \frac{\pi^2}{2} + \frac{\pi^4}{24} - \frac{\pi^6}{720} + \ldots \]

We recall that \(\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \ldots.\)

The series in question is the series for \(\cos x\) with \(x = \pi\); therefore, it converges to \(\cos \pi\), which is \(-1\).

12. (a) Write the complete series equal to \(\int_0^1 e^{-x^2} \, dx\) and show that it converges.

We know \(e^w = 1 + w + \frac{w^2}{2!} + \frac{w^3}{3!} + \ldots\), so by substitution we obtain the following.

\[ e^{-x^2} = 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \cdots = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \ldots \]

Thus, \(\int_0^1 e^{-x^2} \, dx = \int_0^1 \left[ 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \ldots \right] \, dx\)

\[ = \left[ x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \ldots \right]_0^1 \]

\[ = 1 - \frac{1^3}{3} + \frac{1^5}{5 \cdot 2!} - \frac{1^7}{7 \cdot 3!} + \ldots \]

\[ = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k + 1)k!} \]

The terms of this series alternate in sign.

And, \(1 \geq \frac{1^3}{3} \geq \frac{1^5}{5 \cdot 2!} \geq \frac{1^7}{7 \cdot 3!} \geq \ldots \geq 0.\)

And, \(\lim_{k \to \infty} \frac{1}{(2k + 1)k!} = 0.\)

Therefore, by the Alternating Series Test, the series must converge.

(b) If \(f(x) = e^{-x^2}\), what is \(f^{(400)}(0)\)? What is \(f^{(401)}(0)\)?

In the previous part, we found the following.

\[ e^{-x^2} = 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \cdots = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \ldots \]

We know that \(f^{(400)}(0)\) will appear in the Taylor series for \(f(x)\) in the coefficient of the \(x^{400}\) term, which is \(\frac{f^{(400)}(0)}{400!} x^{400}\).

From above, we see that term is \(\frac{x^{400}}{200!}\).

Setting them equal gives \(\frac{f^{(400)}(0)}{400!} x^{400} = \frac{x^{400}}{200!}\), which means that \(f^{(400)}(0) = \frac{400!}{200!}\).
We know that \( f^{(401)}(0) \) will appear in the Taylor series for \( f(x) \) in the coefficient of the \( x^{401} \) term, which is \( \frac{f^{(401)}(0)}{401!} x^{401} \).

However, the Taylor series above for \( f(x) \) has no terms with odd powers of \( x \), meaning that the coefficients of all those terms must be 0.

Therefore, we know that \( f^{(401)}(0) = 0 \).