1. Consider the function \( f(x) = x^6 - 2x^3 \) on the interval \([-2, 2]\).

   (a) Find the \( x \)- and \( y \)-coordinates of any and all local extrema and classify each as a local maximum or local minimum.

   \[ f'(x) = 6x^5 - 6x^2 \]
   \[ 0 = 6x^2(x^3 - 1) \]
   \[ \Rightarrow x = 0, 1 \]

   \[ f \]
   \[
   \begin{array}{c|c|c}
   -2 \leq x < 0 & 0 < x < 1 & 1 < x \leq 2 \\
   \hline
   f' & negative & negative & positive \\
   \hline
   \end{array}
   \]

   \[ y \text{-values: } f(0) = 0, f(1) = -1 \]
   So, \( f \) has a local minimum at \((1, -1)\); \((0, 0)\) is not a local extremum.

   (b) Find the \( x \)- and \( y \)-coordinates of any and all global extrema and classify each as a global maximum or global minimum.

   We check the \( y \)-values at the local extrema and the endpoints.

   \[ y \text{-values: } f(-2) = 80, f(1) = -1, f(2) = 48 \]
   So, \( f \) has a global minimum at \((1, -1)\) and a global maximum at \((-2, 80)\).

   (c) Find the \( x \)-coordinate(s) of any and all inflection points.

   \[ f''(x) = 30x^4 - 12x \]
   \[ 0 = 6x(5x^3 - 2) \]
   \[ \Rightarrow x = 0, \sqrt[3]{0.4} \]

   \[ f' \]
   \[
   \begin{array}{c|c|c}
   x < 0 & 0 < x < \sqrt[3]{0.4} & \sqrt[3]{0.4} < x \\
   \hline
   f' & positive & negative & positive \\
   \hline
   \end{array}
   \]

   So, the \( x \)-values of the inflection points of \( f \) are \( x = 0 \) and \( x = \sqrt[3]{0.4} \).

2. Your company is mass-producing a cylindrical container. The flat portion (top and bottom) costs 3 cents per square inch and the curved (lateral) portion costs 5 cents per square inch. If your budget is \$9.00 per container, what dimensions will give the largest volume?

   area of circle = \( \pi r^2 \)  
   lateral area of cylinder = \( 2\pi rh \)  
   volume of cylinder = \( \pi r^2 h \)

   Objective function: volume = \( V = \pi r^2 h \)

   We need to get this down to a function of just one variable, so we use the constraint equation: cost = \( 900 = 3 \cdot 2 \cdot \pi r^2 + 5 \cdot 2\pi rh \)

   \[ 900 = 6\pi r^2 + 10\pi rh \]
   \[ 900 - 6\pi r^2 = 10\pi rh \]
   \[ \frac{900 - 6\pi r^2}{10\pi r} = h \]

   Substituting this back into the objective function gives
\[ V = \pi r^2 h = \pi r^2 \cdot \frac{900 - 6\pi r^2}{10\pi r} = r \cdot \frac{900 - 6\pi r^2}{10} = \frac{1}{10} (900r - 6\pi r^3). \]

Now that we have \( V \) as a function of just one variable, we find its maximum.

\[ V'(x) = \frac{1}{10} (900 - 18\pi r^2) \]

\[ 0 = \frac{1}{10} (900 - 18\pi r^2) \]

\[ \Rightarrow 18\pi r^2 = 900 \]

\[ \Rightarrow r^2 = \frac{50}{\pi} \]

\[ \Rightarrow r = \sqrt{\frac{50}{\pi}} \]

\[
\begin{array}{c|cc}
0 < x < \frac{\sqrt{50}/\pi}{\sqrt{50}/\pi} & \frac{\sqrt{50}/\pi}{\sqrt{50}/\pi} < x \\
\hline
f' & positive & negative \\
\end{array}
\]

Thus, we have in fact found the global maximum at \( r = \sqrt{50/\pi}. \)

And \( h = \frac{900 - 6\pi r^2}{10\pi r} = \ldots \text{much simplifying} \ldots = \sqrt{\frac{72}{\pi}} \approx 4.787 \) inches.

3. You are standing on a pier, 6 feet above the deck of a boat. Attached to the boat is a line, which you are pulling in at a rate of 3 feet per second. When there are 10 feet of line between your hand and the boat, at what rate is the boat moving across the water?

So, we write an equation that relates \( a \) and \( b \) and then differentiate implicitly with respect to time \( t. \)

\[ a^2 + 6^2 = b^2 \]

\[ 2a \frac{da}{dt} + 0 = 2b \frac{db}{dt} \]

\[ \frac{da}{dt} = \frac{b}{a} \frac{db}{dt} \]

At the moment in question, \( b = 10, a = 8 \) (by the Pythagorean Theorem), and \( \frac{db}{dt} = -3. \)

So, \( \frac{da}{dt} = \frac{10}{8} \cdot (-3) = -3.75 \) feet per second, meaning the boat is moving toward the dock at 3.75 feet per second.

4. Use the Intermediate Value Theorem to show that \( f(x) = x^3 - 2x - 1 \) has a root on \([1,2].\)

IVT: If \( f \) is continuous on \([a,b]\) and \( y \) is a number between \( f(a) \) and \( f(b), \) then there is a number \( c \) between \( a \) and \( b \) such that \( f(c) = y. \)

For the function given above, \( f(1) = -2 \) and \( f(2) = 3. \) Since 0 is a number between \(-2\) and 3, the IVT says there is a number \( c \) between 1 and 2 such that \( f(c) = 0; \) this \( c \) is the desired root.

5. What (if anything) does the Extreme Value Theorem say about \( f(x) = x^2 \) on each of the following intervals?

EVT: If \( f \) is continuous on \([a,b], \) then \( f \) has both a maximum and a minimum on \([a,b]. \)
6. Find the value of the constant $c$ that the Mean Value Theorem specifies for $f(x) = x^3 + x$ on $[0, 3]$.

MVT: If $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$, then there is a number $c$ between $a$ and $b$ such that
\[
  f'(c) = \frac{f(b) - f(a)}{b - a}.
\]

For our function, we have
\[
  \frac{f(3) - f(0)}{3 - 0} = \frac{30 - 0}{3} = 10.
\]

And $f'(x) = 3x^2 + 1$, so $f'(c) = 3c^2 + 1$.

So, we solve $3c^2 + 1 = 10$, which means $c = \sqrt{3}$. (The other solution, $x = -\sqrt{3}$, is not in our interval $[0, 3]$.)

7. Water is leaking out of a tank at a decreasing rate $r(t)$ as shown below.

<table>
<thead>
<tr>
<th>time (min)</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>rate (gal/min)</td>
<td>15</td>
<td>11</td>
<td>8</td>
<td>4</td>
<td>3</td>
</tr>
</tbody>
</table>

(a) Find an overestimate and underestimate for the total amount that leaked out during these 8 minutes.

overestimate $= L_4 = (15 + 11 + 8 + 4)(2) = 76$

underestimate $= R_4 = (11 + 8 + 4 + 3)(2) = 52$

(b) Interpret the expression $\int_{2}^{6} r(t) dt$ in terms of the situation described above.

This integral gives the amount (in gallons) of water that leaked from the tank on the interval $[2, 6]$ minutes.

8. Consider the graph of $f(t)$ shown. It is made of straight lines and a semicircle.

Let $G(x) = \int_{0}^{x} f(t) dt$ and $H(x) = \int_{-3}^{x} f(t) dt$.

(a) Compute $G(2)$, $G(4)$, and $H(4)$.

$G(2)$ is the area under $f$ between $t = 0$ and $t = 2$. This is a rectangle plus a triangle and has area
\[
  2 \cdot 1 + \frac{1}{2} \cdot 2 \cdot 1 = 3.
\]

Similarly, $G(4) = 2 \cdot 1 + \frac{1}{2} \cdot 2 \cdot 1 + \frac{1}{2} \pi (1)^2 = 3 + \frac{\pi}{2}$.
\( H(4) \) is the area under \( f \) between \( t = -3 \) and \( t = 4 \). Remember that area below the \( t \)-axis counts as negative.

\[
H(4) = - (2 \cdot 1 + \frac{1}{2} \cdot 2 \cdot 1) + \frac{1}{2} \cdot 2 \cdot 1 + [\text{area under } f \text{ from } 0 \text{ to } 4, \text{ found above as } G(4)] \\
= - 2 + \left[ 3 + \frac{\pi}{2} \right] \\
= 1 + \frac{\pi}{2}
\]

(b) Where is \( G \) increasing? Where is \( G \) decreasing?

For parts (b), (c), and (d), recall that we learned in class that \( G' = f \).

\( G \) is increasing where \( f \) is positive: \((-1, 4]\). Note that \( G \) has a horizontal slope at \( x = 2 \) but since \( f \) is positive on each side of \( t = 2 \), we say \( G \) is increasing at \( x = 2 \).

\( G \) is decreasing where \( f \) is negative: \((-\infty, -1]\).

(c) Where is \( G \) concave up? Where is \( G \) concave down?

\( G \) is concave up where \( f \) is increasing: \((-2, 0) \cup (2, 3)\).

\( G \) is concave down where \( f \) is decreasing: \((1, 2) \cup (3, 4]\).

(d) At what \( x \)-value(s) does \( G \) have a local maximum? At what \( x \)-value(s) does \( G \) have a local minimum?

\( G \) has a local maximum where \( f \) changes from positive to negative: never.

\( G \) has a local minimum where \( f \) changes from negative to positive: \( x = -1 \).

(e) Find a formula that relates \( G \) and \( H \).

From their definitions, \( H(x) = \int_{-3}^{0} f(t) \, dt + G(x) = -2 + G(x) \).

(f) How would your answers to (b), (c), and (d) change if the questions were about \( H \) instead of \( G \)?

They would not change at all because \( H'(x) = G'(x) \).

9. (a) Use sigma notation to express \( L_{10} \) and \( M_{10} \) as approximations to \( \int_{20}^{60} \ln x \, dx \).

We’re subdividing the interval into 10 pieces, so each piece has width \( \Delta x = \frac{60 - 20}{10} = 4 \).

\[
L_{10} = [f(20) + f(24) + f(28) + ... + f(52) + f(56)] \Delta x \\
= [\ln(20) + \ln(24) + \ln(28) + ... + \ln(52) + \ln(56)] \cdot 4 \\
= \sum_{k=0}^{9} \ln(20 + 4k) \cdot 4
\]

\[
M_{10} = [f(22) + f(26) + f(30) + ... + f(54) + f(58)] \Delta x \\
= [\ln(22) + \ln(26) + \ln(30) + ... + \ln(54) + \ln(58)] \cdot 4 \\
= \sum_{k=0}^{9} \ln(22 + 4k) \cdot 4
\]

(b) Draw a sketch that represents the sum \( M_{4} \).

Now we’re subdividing the interval into 4 pieces, so each piece has width \( \Delta x = \frac{60 - 20}{4} = 10 \).

Note that the height of each rectangle is determined by the \( y \)-value of the curve at the middle \( x \)-value of the rectangle (that is, at \( x = 25, 35, 45, 55 \)).
10. Find the following.

(a) all antiderivatives of \(1 + 2x + x^3 + 4\sqrt{x} + \frac{1}{x^5}\)

Any such antiderivative will take the form \(x + x^2 + \frac{x^4}{4} + \frac{x^{3/2}}{3/2} + \frac{x^{-4}}{-4} + C\).

Note that we have used the facts that \(\sqrt{x} = x^{1/2}\) and \(1/x^5 = x^{-5}\).

(b) \(\int_{-2}^{2} \sqrt{4 - x^2} \, dx = \frac{1}{2} \pi (2)^2 = 2\pi\) \hspace{1cm} This integral represents the area of a semicircle of radius 2.

(c) \(\frac{d}{dx} \int_{1}^{x} \sin \sqrt{t} \, dt = \sin \sqrt{x}\) \hspace{1cm} The derivative of the area function is the original function.
(d) \[ \int_0^2 x^2 \, dx \]

Do this first with the limit definition of the definite integral then check your answer with the Fundamental Theorem.

You may use the fact that \( \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \).

We will do this with a right-hand sum \( R_n \).

We subdivide \([0, 2]\) into \( n \) equal pieces, each of width \( \Delta x = \frac{2 - 0}{n} = \frac{2}{n} \).

Thus, \( x_1 = \frac{2}{n} \), \( x_2 = \frac{4}{n} \), \( x_3 = \frac{6}{n} \), ..., and \( x_n = \frac{2n}{n} \).

\[
\int_0^2 x^2 \, dx = \lim_{n \to \infty} R_n
\]

This is our limit definition of the definite integral.

\[
= \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k) \Delta x
\]

This is our definition of a right-hand sum.

\[
= \lim_{n \to \infty} \sum_{k=1}^{n} \left( \frac{2k}{n} \right)^2 \frac{2}{n}
\]

From above, \( x_k = \frac{2k}{n} \) and \( \Delta x = \frac{2}{n} \).

\[
= \lim_{n \to \infty} \sum_{k=1}^{n} \left( \frac{4k^2}{n^2} \right) \frac{2}{n}
\]

Our function is \( f(x) = x^2 \).

\[
= \lim_{n \to \infty} \frac{8}{n^3} \sum_{k=1}^{n} k^2
\]

We can pull out \( \frac{8}{n^3} \) because it doesn’t depend on \( k \).

\[
= \lim_{n \to \infty} \frac{8}{n^3} \frac{n(n+1)(2n+1)}{6}
\]

We apply the handy fact we were given above.

\[
= \lim_{n \to \infty} \frac{4}{n^2} \frac{(n+1)(2n+1)}{3}
\]

\[
= \lim_{n \to \infty} \frac{4}{3} \frac{2n^2 + 3n + 1}{n^2}
\]

\[
= \lim_{n \to \infty} \frac{4}{3} \left( 2 + \frac{3}{n} + \frac{1}{n^2} \right)
\]

\[
= \lim_{n \to \infty} \frac{4}{3} \left( 2 + 0 + 0 \right)
\]

\[
= \frac{8}{3}
\]

Now check with the FTC: \( \int_0^2 x^2 \, dx = \frac{x^3}{3} \bigg|_0^2 = \frac{2^3}{3} - \frac{0^3}{3} = \frac{8}{3} \). That was slightly easier.