1. Consider the function \( f(x) = \frac{3}{5-2x} \).

(a) Is this function continuous on the interval \((-\infty, \infty)\)? Explain.
   No. The function is discontinuous at \( x = 2.5 \), where \( f \) is undefined (and has a vertical asymptote).
(b) Compute the average rate of change of \( f \) on \([2, 2.01]\).
   \[
   \frac{f(2.01) - f(2)}{2.01 - 2} = \left[ \frac{3}{5-2(2.01)} - \frac{3}{5-2(2)} \right] \cdot \frac{1}{.01} \approx 6.122
   \]

(c) Using the limit definition of the derivative, compute \( f'(x) \).
   \[
   f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{provided this limit exists}
   \]
   \[
   = \lim_{h \to 0} \frac{3}{5-2(x+h)} - \frac{3}{5-2x} \quad \text{common denominator}
   \]
   \[
   = \lim_{h \to 0} \frac{3(5-2x) - 3(5-2(x+h))}{h(5-2(x+h))(5-2x)}
   \]
   \[
   = \lim_{h \to 0} \frac{15 - 6x - (15 - 6x - 6h)}{h(5-2(x+h))(5-2x)}
   \]
   \[
   = \lim_{h \to 0} \frac{6h}{h(5-2(x+h))(5-2x)}
   \]
   \[
   = \lim_{h \to 0} \frac{6}{5-2(x+h))(5-2x)}
   \]
   \[
   = \frac{6}{(5-2x)^2}
   \]

(d) Find the equation of the tangent line to \( f \) at \( x = 2 \).
   We want \( y = mx + b \). \( m = f'(2) = \frac{6}{(5-2(2))^2} = 6 \), so \( y = 6x + b \).
   [Note that this slope agrees well with our answer from (b) above.]
   When \( x = 2 \), \( y = f(2) = \frac{3}{5-2(2)} = 3 \).
   Thus, \( 3 = 6 \cdot 2 + b \), so \( b = -9 \) and we have \( y = 6x - 9 \).

2. Given that \( f(0) = 2, g(0) = 3, f'(0) = 5, g'(0) = 7 \), and \( f'(3) = \pi \) compute the following.

(a) \( h'(0) \) if \( h(z) = f(z)g(z) \)
   \( h'(0) = f'(0)g(0) + f(0)g'(0) = (5)(3) + (2)(7) = 29 \)

(b) \( j'(0) \) if \( j(z) = \frac{f(z)}{g(z)} \)
   \( j'(0) = \frac{f'(0)g(0) - f(0)g'(0)}{[g(0)]^2} = \frac{(5)(3) - (2)(7)}{3^2} = \frac{1}{9} \)

(c) \( k'(0) \) if \( k(z) = f(g(z)) \)
   \( k'(0) = f'(g(0)) \cdot g'(0) = f'(3) \cdot (7) = (\pi)(7) = 7\pi \)
3. (a) Find \( \frac{dy}{dt} \) if \( y = t^6 + 5t + e^t + \frac{t}{5} + \frac{5}{\sqrt{t}} + \ln(5t) + \arctan(5t) + \ln(5) + \sin(5) \).

\[
\frac{dy}{dt} = 5t^5 + (\ln 5)5^t + 0 + \frac{1}{5} - 5t^{-2} + 5 \cdot \frac{-1}{5}t^{-6/5} + \frac{1}{5t} \cdot 5 + \frac{1}{1 + (5t)^2} \cdot 5 + 0 + 0
\]

\[
= 5t^4 + (\ln 5)5^t + \frac{1}{5} - \frac{1}{t^2} - \frac{1}{t^{5/5}} + \frac{1}{t} + \frac{5}{1 + 25t^2}
\]

(b) Find \( \frac{dy}{dx} \) if \( y = \sqrt{x} \cos(7x^3) \).

\[
\frac{dy}{dx} = \frac{1}{3}x^{-2/3} \cos(7x^3) + \sqrt{x}(-\sin(7x^3)(21x^2)) = \frac{\cos(7x^3)}{3x^{2/3}} - 21x^{7/3} \sin(7x^3)
\]

(c) Find \( \frac{dy}{dz} \) if \( y = \frac{\tan^4 z}{\tan 4 - 7z} \).

\[
\frac{dy}{dz} = \frac{\tan^4 z}{\tan 4 - 7z^2} \cdot \frac{(\tan 4 - 7)(\tan^2 z + 1)}{\tan^2 z}
\]

(d) Find \( \frac{dy}{dr} \) if \( y = \tan(e^{r^2 \arcsin(5r)}) \).

\[
\frac{dy}{dr} = \sec^2(e^{r^2 \arcsin(5r)}) \cdot e^{r^2 \arcsin(5r)} \cdot \left[ r^2 \frac{1}{\sqrt{1 - 25r^2}} \cdot 5 + 2r \arcsin(5r) \right]
\]

(e) Find \( \frac{dy}{dx} \) if \( y^3 + yx^2 + x^2 = 3y^2 \).

Here we use implicit differentiation.

\[
3y^2 \frac{dy}{dx} + \frac{dy}{dx} x^2 + 2xy + 2x = 6y \frac{dy}{dx}
\]

\[
3y^2 \frac{dy}{dx} + \frac{dy}{dx} x^2 - 6y \frac{dy}{dx} = -2xy - 2x
\]

\[
\frac{dy}{dx} = \frac{-2xy - 2x}{3y^2 + x^2 - 6y}
\]

(f) Find \( \frac{dy}{dx} \) if \( y = (1 + x^6)^{8x} \). Since we have \( x \) in the base and the exponent, we need logarithmic differentiation.

\[
\ln y = 8x \ln(1 + x^6)
\]

\[
\frac{1}{y} \frac{dy}{dx} = 8 \cdot \ln(1 + x^6) + 8x \cdot \frac{1}{1 + x^6} \cdot 6x^5
\]

\[
\frac{dy}{dx} = \left[ 8 \cdot \ln(1 + x^6) + \frac{48x^6}{1 + x^6} \right] \cdot y
\]

\[
\frac{dy}{dx} = \left[ 8 \cdot \ln(1 + x^6) + \frac{48x^6}{1 + x^6} \right] \cdot (1 + x^6)^{8x}
\]
4. Given the graph of $f$, sketch a graph of $f'$ and a graph of $F$, an antiderivative of $f$ such that $F(0) = -1$.

Note: The concave up portion on the left side of the graph of $f$ is a perfect parabola, so its derivative $(f')$ is linear; since you don’t know the equation for $f$, your graph of $f'$ may be concave up/down there.

5. Shown below is a graph of $f'$ on its entire domain. The graph is NOT $f$.

At which $x$-value(s)

(a) does $f$ have a stationary point? $c, f, h$
(b) $f$ decreasing? $(c, f) \cup (h, j)$
(c) $f'$ increasing? $[a, b) \cup (d, g) \cup (i, j]$
(d) $f'$ decreasing? $(b, d) \cup (g, i)$
(e) $f$ concave up? $[a, b) \cup (d, g) \cup (i, j]$
(f) $f$ concave down? $(b, d) \cup (g, i)$

(b) does $f$ have a local max? $c, h$
(c) does $f$ have a local min? $f$
(d) does $f'$ have a stationary point? $b, d, g, i$
(e) does $f'$ have a local max? $b, g$
(f) does $f'$ have a local min? $d, i$
(g) is $f$ greatest? $c$
(h) is $f$ least? $j$
(i) is $f'$ greatest? $b$
(j) is $f'$ least? $d$
(k) is $f''$ greatest? $e$
(l) is $f''$ least? $c$

On what interval(s) is

(a) $f$ increasing? $[a, c) \cup (f, h)$
6. Is \( y = 7e^{3x} \) a solution to the differential equation \( y'' + 2y' - 15y = 0 \)? Explain.

A given function \( y \) will be a solution to the differential equation if, when we substitute in \( y'' \), \( y' \), and \( y \), the equation is satisfied (that is, both sides of it are equal).

Since \( y = 7e^{3x} \), we know that \( y' = 21e^{3x} \) and \( y'' = 63e^{3x} \) from the Chain Rule.

Now we check to see whether our \( y \) satisfies the differential equation.

\[
y'' + 2y' - 15y = 0
\]
\[
63e^{3x} + 2 \cdot 21e^{3x} - 15 \cdot 7e^{3x} = 0
\]
\[
63e^{3x} + 42e^{3x} - 105e^{3x} = 0
\]
\[
0 = 0
\]

So, we see that \( y = 7e^{3x} \) is in fact a solution to this differential equation.

7. Rewrite \( \sin(\arctan(5x)) \) as an algebraic expression. [Students in the 8:00, 9:30, and 1:10 sections may omit this problem.]

Let \( \theta = \arctan(5x) \). That is, \( \theta \) is the angle whose tangent is \( 5x \).

We draw a triangle for which \( \frac{\text{opposite}}{\text{adjacent}} = \frac{5x}{1} = 5x \).

\[
\begin{align*}
\theta & \quad 5x \\
1 & \quad z
\end{align*}
\]

\[
1^2 + (5x)^2 = z^2 \Rightarrow z = \sqrt{1 + 25x^2}
\]

\[
\sin(\arctan(5x)) = \sin \theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{5x}{\sqrt{1 + 25x^2}}
\]

8. Evaluate the following limits.

Throughout this solution, the symbol ★ will stand for whatever notation your instructor prefers for using L’Hopital’s Rule on the indeterminate form 0/0; this may be \( = \) or \( \not= \) or \( = \) or \( = \) or = or = “0/0” or “has the form \( \frac{0}{0} \)” and so, by L’Hopital’s Rule, is equal to” or something else. The symbol ⌂ will serve the same purpose for the indeterminate form \( \infty/\infty \).

(a) \[
\lim_{x \to \infty} \frac{x^2}{\ln x} \quad \text{\( \because \)} \quad \lim_{x \to \infty} \frac{2x}{1/x} = \lim_{x \to \infty} 2x^2 = \infty
\]

(b) \[
\lim_{z \to 0} \frac{\sin(12z) - 12z}{2z} \quad \text{★} \quad \lim_{z \to 0} \frac{12 \cos(12z) - 12}{3z^2} \quad \text{★} \quad \lim_{z \to 0} \frac{-144 \sin(12z)}{6z} \quad \text{★} \quad \lim_{z \to 0} \frac{-1728 \cos(12z)}{6} = -288
\]

(c) \[
\lim_{x \to 0} \frac{e^x - 1}{\cos x} = \frac{0}{1} = 0
\]

(d) \[
\lim_{r \to 2} \frac{r^3 - 8}{r - 2} \quad \text{★} \quad \lim_{r \to 2} \frac{3r^2}{1} = 12
\]

(e) \[
\lim_{x \to 0^+} x^3 \ln x \quad \text{[Students in the 8:00 and 9:30 sections may omit this problem.]}
\]

This is of the form \( 0 \cdot (-\infty) \), so we rewrite it as a fraction to turn it into a L’Hopital’s Rule problem.

\[
\lim_{x \to 0^+} x^3 \ln x = \lim_{x \to 0^+} \frac{\ln x}{\frac{1}{x^3}} \quad \text{\( \because \)} \quad \lim_{x \to 0^+} \frac{\ln x}{x^{-3}} = \lim_{x \to 0^+} \frac{1}{x} \cdot x^4 = \lim_{x \to 0^+} \frac{x^3}{-3} = 0 = 0
\]