1. Your company is mass-producing a cylindrical container. The flat portion (top and bottom) costs 3 cents per square inch and the curved (lateral) portion costs 5 cents per square inch. If your budget is $9.00 per container, what dimensions will give the largest volume?

**area of circle = \( \pi r^2 \)  
**lateral area of cylinder = \( 2\pi rh \)  
**volume of cylinder = \( \pi r^2 h \)**

**Objective function:** \( V = \pi r^2 h \)

We need to get this down to a function of just one variable, so we use the

**constraint equation:** \( \text{cost} = 900 = 3 \cdot 2 \cdot \pi r^2 + 5 \cdot 2\pi rh \)

\[
900 = 6\pi r^2 + 10\pi rh \\
900 - 6\pi r^2 = 10\pi rh \\
\frac{900 - 6\pi r^2}{10\pi r} = h 
\]

Substituting this back into the objective function gives

\[
V = \pi r^2 h = \pi r^2 \cdot \frac{900 - 6\pi r^2}{10\pi r} = r \cdot \frac{900 - 6\pi r^2}{10} = \frac{1}{10} (900r - 6\pi r^3) 
\]

Now that we have \( V \) as a function of just one variable, we find its maximum.

\[
V'(x) = \frac{1}{10} (900 - 18\pi r^2) \\
0 = \frac{1}{10} (900 - 18\pi r^2) \\
\Rightarrow 18\pi r^2 = 900 \\
\Rightarrow r^2 = \frac{50}{\pi} \\
\Rightarrow r = \sqrt{\frac{50}{\pi}} 
\]

<table>
<thead>
<tr>
<th>( f' )</th>
<th>( 0 &lt; x &lt; \sqrt{\frac{50}{\pi}} )</th>
<th>( \sqrt{\frac{50}{\pi}} &lt; x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>positive</td>
<td></td>
<td>negative</td>
</tr>
</tbody>
</table>

Thus, we have in fact found the global maximum at \( r = \sqrt{\frac{50}{\pi}} \).

And \( h = \frac{900 - 6\pi r^2}{10\pi r} = ...\text{much simplifying...} = \frac{\sqrt{72}}{\pi} \approx 4.787 \) inches.

2. You are standing on a pier, 6 feet above the deck of a boat. Attached to the boat is a line, which you are pulling in at a rate of 3 feet per second. When there are 10 feet of line between your hand and the boat, at what rate is the boat moving across the water?

![Diagram](image-url)

We know \( \frac{db}{dt} \), and we want to find \( \frac{da}{dt} \).
So, we write an equation that relates $a$ and $b$ and then differentiate implicitly with respect to time $t$.

$$a^2 + b^2 = b^2$$

$$2a \frac{da}{dt} + 0 = 2b \frac{db}{dt}$$

$$\frac{da}{dt} = \frac{b \, db}{a \, dt}$$

At the moment in question, $b = 10$, $a = 8$ (by the Pythagorean Theorem), and $\frac{db}{dt} = -3$.

So, $\frac{da}{dt} = \frac{10}{8} \cdot (-3) = -3.75$ feet per second, meaning the boat is moving toward the dock at 3.75 feet per second.

3. Use the Intermediate Value Theorem to show that $f(x) = x^3 - 2x - 1$ has a root on $[1, 2]$.

IVT: If $f$ is continuous on $[a, b]$ and $y$ is a number between $f(a)$ and $f(b)$, then there is a number $c$ between $a$ and $b$ such that $f(c) = y$.

For the function given above, $f(1) = -2$ and $f(2) = 3$. Since 0 is a number between $-2$ and 3, the IVT says there is a number $c$ between 1 and 2 such that $f(c) = 0$; this $c$ is the desired root.

4. What (if anything) does the Extreme Value Theorem say about $f(x) = x^2$ on each of the following intervals?

EVT: If $f$ is continuous on $[a, b]$, then $f$ has both a maximum and a minimum on $[a, b]$.

(a) $[1, 4]$

$f$ has a maximum and a minimum on $[1, 4]$

(b) $(1, 4)$

The EVT doesn’t apply because $(1, 4)$ is not a closed interval since its endpoints are not included.

5. Find the value of the constant $c$ that the Mean Value Theorem specifies for $f(x) = x^3 + x$ on $[0, 3]$.

MVT: If $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$, then there is a number $c$ between $a$ and $b$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

For our function, we have $\frac{f(3) - f(0)}{3 - 0} = \frac{30 - 0}{3} = 10$.

And $f'(x) = 3x^2 + 1$, so $f'(c) = 3c^2 + 1$.

So, we solve $3c^2 + 1 = 10$, which means $c = \sqrt{3}$. (The other solution, $x = -\sqrt{3}$, is not in our interval $[0, 3]$.)

6. Water is leaking out of a tank at a decreasing rate $r(t)$ as shown below.

<table>
<thead>
<tr>
<th>time (min)</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>rate (gal/min)</td>
<td>15</td>
<td>11</td>
<td>8</td>
<td>4</td>
<td>3</td>
</tr>
</tbody>
</table>

(a) Find an overestimate and underestimate for the total amount that leaked out during these 8 minutes.

overestimate $= L_4 = (15 + 11 + 8 + 4)(2) = 76$

underestimate $= R_4 = (11 + 8 + 4 + 3)(2) = 52$

(b) Interpret the expression $\int_2^6 r(t) \, dt$ in terms of the situation described above.

This integral gives the amount (in gallons) of water that leaked from the tank on the interval $[2, 6]$ minutes.
7. Consider the graph of \( f(t) \) shown. It is made of straight lines and a semicircle.

![Graph of f(t)](image)

Let \( G(x) = \int_0^x f(t) \, dt \) and \( H(x) = \int_0^x f(t) \, dt \).

(a) Compute \( G(2), G(4), G(-4), \) and \( H(4) \).

First, \( G(2) = \int_0^2 f(t) \, dt \) is the area under \( f \) between \( t = 0 \) and \( t = 2 \). This is a rectangle plus a triangle and has area \( 2 \cdot 1 + \frac{1}{2} \cdot 2 \cdot 1 = 3 \).

Similarly, \( G(4) = \int_0^4 f(t) \, dt = 2 \cdot 1 + \frac{1}{2} \cdot 2 \cdot 1 + \frac{1}{2} \pi (1)^2 = 3 + \frac{\pi}{2} \).

Now, remembering that area below the \( t \)-axis counts as negative and that \( \int_b^a f(t) \, dt = -\int_a^b f(t) \, dt \), we have
\[
G(-4) = \int_0^{-4} f(t) \, dt = -\int_{-4}^0 f(t) \, dt = -\left[ -2 \cdot 2 - \frac{1}{2} \cdot 1 \cdot 2 + \frac{1}{2} \cdot 1 \cdot 2 \right] = 4.
\]

Finally, \( H(4) = \int_{-3}^4 f(t) \, dt = -2 \cdot 1 - \frac{1}{2} \cdot 1 \cdot 2 + \frac{1}{2} \cdot 1 \cdot 2 + 2 \cdot 1 + \frac{1}{2} \cdot 2 \cdot 1 + \frac{1}{2} \pi (1)^2 = 1 + \frac{\pi}{2} \).

(b) Where is \( G \) increasing? Where is \( G \) decreasing?

For parts (b), (c), and (d), recall that we learned in class that \( G' = f \).

\( G \) is increasing where \( f \) is positive: \((-1, 4)\). Note that \( G \) has a horizontal slope at \( x = 2 \) but since \( f \) is positive on each side of \( t = 2 \), we say \( G \) is increasing at \( x = 2 \).

\( G \) is decreasing where \( f \) is negative: \((-4, -1)\).

(c) Where is \( G \) concave up? Where is \( G \) concave down?

\( G \) is concave up where \( f \) is increasing: \((-2, 0) \cup (2, 3)\).

\( G \) is concave down where \( f \) is decreasing: \((1, 2) \cup (3, 4)\).

(d) At what \( x \)-value(s) does \( G \) have a local maximum? At what \( x \)-value(s) does \( G \) have a local minimum?

\( G \) has a local maximum where \( f \) changes from positive to negative: \( x = 2 \).

\( G \) has a local minimum where \( f \) changes from negative to positive: \( x = -1 \).

(e) Find a formula that relates \( G \) and \( H \).

From their definitions, \( H(x) = \int_{-3}^x f(t) \, dt + G(x) = -2 + G(x) \).

(f) How would your answers to (b), (c), and (d) change if the questions were about \( H \) instead of \( G \)?

They would not change at all because \( H'(x) = G'(x) \).

8. (a) Use sigma notation to express \( L_{10} \) and \( M_{10} \) as approximations to \( \int_{20}^{60} \ln x \, dx \).
We’re subdividing the interval into 10 pieces, so each piece has width \( \Delta x = \frac{60 - 20}{10} = 4 \).

\[
L_{10} = [f(20) + f(24) + f(28) + \ldots + f(52) + f(56)]\Delta x \\
= [\ln(20) + \ln(24) + \ln(28) + \ldots + \ln(52) + \ln(56)] \cdot 4 \\
= \sum_{k=0}^{9} \ln(20 + 4k) \cdot 4
\]

\[
M_{10} = [f(22) + f(26) + f(30) + \ldots + f(54) + f(58)]\Delta x \\
= [\ln(22) + \ln(26) + \ln(30) + \ldots + \ln(54) + \ln(58)] \cdot 4 \\
= \sum_{k=0}^{9} \ln(22 + 4k) \cdot 4
\]

(b) **Draw a sketch that represents the sum \( M_4 \).**

Now we’re subdividing the interval into 4 pieces, so each piece has width \( \Delta x = \frac{60 - 20}{4} = 10 \).

Note that the height of each rectangle is determined by the \( y \)-value of the curve at the middle \( x \)-value of the rectangle (that is, at \( x = 25, 35, 45, 55 \)).

9. Find the following.

(a) **all antiderivatives of** \( 1 + 2x + x^3 + 4\sqrt{x} + \frac{1}{x^2} + \sec^2(6x) + \frac{7}{1 + 100x^2} \)

Any such antiderivative will take the form \( x + x^2 + \frac{x^4}{4} + 4x^{3/2} + \frac{x^{-4}}{-4} + \frac{\tan(6x)}{6} + \frac{7\arctan(10x)}{10} + C \).

Note that we have used the facts that \( \sqrt{x} = x^{1/2} \) and \( 1/x^5 = x^{-5} \).

(b) \[
\int_{-2}^{2} \sqrt{4 - x^2} \, dx = \frac{1}{2} \pi (2)^2 = 2\pi \quad \text{This integral represents the area of a semicircle of radius 2.}
\]

(c) \[
\frac{d}{dx} \int_{1}^{x} \sqrt{t} \, dt = \sin \sqrt{x} \quad \text{The derivative of the area function is the original function.}
\]
(d) $\int_0^2 x^2 \, dx$

Do this first with the limit definition of the definite integral then check your answer with the Fundamental Theorem.

You may use the fact that $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.

We will do this with a right-hand sum $R_n$.

We subdivide $[0, 2]$ into $n$ equal pieces, each of width $\Delta x = \frac{2 - 0}{n} = \frac{2}{n}$.

Thus, $x_1 = \frac{2}{n}$, $x_2 = \frac{4}{n}$, $x_3 = \frac{6}{n}$, ..., and $x_n = \frac{2n}{n}$.

\[
\int_0^2 x^2 \, dx = \lim_{n \to \infty} R_n
\]

This is our limit definition of the definite integral.

\[
= \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k) \Delta x
\]

This is our definition of a right-hand sum.

\[
= \lim_{n \to \infty} \sum_{k=1}^{n} \left( \frac{2k}{n} \right)^2 \frac{2}{n}
\]

From above, $x_k = \frac{2k}{n}$ and $\Delta x = \frac{2}{n}$.

\[
= \lim_{n \to \infty} \sum_{k=1}^{n} \left( \frac{4k^2}{n^2} \right) \frac{2}{n}
\]

Our function is $f(x) = x^2$.

\[
= \lim_{n \to \infty} \frac{8}{n^3} \sum_{k=1}^{n} k^2
\]

We can pull out $\frac{8}{n^3}$ because it doesn’t depend on $k$.

\[
= \lim_{n \to \infty} \frac{8}{n^3} \frac{n(n+1)(2n+1)}{6}
\]

We apply the handy fact we were given above.

\[
= \lim_{n \to \infty} \frac{4}{n^2} \frac{(n + 1)(2n + 1)}{3}
\]

\[
= \lim_{n \to \infty} \frac{4}{3} \frac{2n^2 + 3n + 1}{n^2}
\]

\[
= \lim_{n \to \infty} \frac{4}{3} \left( 2 + \frac{3}{n} + \frac{1}{n^2} \right)
\]

\[
= \lim_{n \to \infty} \frac{4}{3} (2 + 0 + 0)
\]

\[
= \frac{8}{3}
\]

Now check with the FTC: $\int_0^2 x^2 \, dx = \frac{x^3}{3} \bigg|_0^2 = \frac{2^3}{3} - \frac{0^3}{3} = \frac{8}{3}$. That was slightly easier.