NAME:

Show ALL your work CAREFULLY.

(a) Find the interval of convergence of the following power series [Do not forget to check the endpoints!].

\[ \sum_{n=1}^{\infty} \frac{2^n}{3n} (x + 3)^n \]

Applying the Ratio Test to the corresponding series of absolute values, we have

\[ L = \lim_{n \to \infty} \frac{2^{n+1}}{3(n+1)} \frac{|x + 3|^{n+1}}{2^n |x + 3|^n} \]

\[ = \lim_{n \to \infty} \frac{2}{n+1} \cdot |x + 3| = 2|x + 3| \]

If \(2|x + 3| < 1\) then the original power series converges and it may converge at the endpoints. Note that \(2|x + 3| < 1\) is equivalent to \(|x + 3| < \frac{1}{2}\) which in turn is equivalent to \(-\frac{7}{2} < x < -\frac{5}{2}\).

At \(x = -\frac{5}{2}\), the series is \(\sum_{n=1}^{\infty} \frac{2^n}{3n} \left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{1}{3n}\) and it diverges since it is one-third of the harmonic series.

At \(x = -\frac{7}{2}\), the series is \(\sum_{n=1}^{\infty} \frac{2^n}{3n} \left(-\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{3n}\) and it converges since it is negative one-third of the alternating harmonic series.

Thus, we conclude that the interval of convergence is \(-\frac{7}{2} \leq x < \frac{5}{2}\).

(b) Find a power series representation of \(\sin \frac{x}{x}\). State the interval of convergence.

Recall that

\[ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \quad \text{for all } x. \]

If \(x \neq 0\) then

\[ \frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots \quad \text{for all } x \text{ except } x = 0. \]

Hence, the interval of convergence is \(-\infty < x < 0 \cup 0 < x < \infty\).