Consider the following infinite series
\[ \sum_{n=1}^{\infty} \left( -1 \right)^{n+1} \left( \frac{n}{n^2 + 2} \right). \]

(a) Use the Alternating Series Test (A.S.T.) to show that the series converges. Be sure to show HOW you use the A.S.T.

In order to apply the A.S.T., we need to show (i) \( c_1 \geq c_2 \geq c_3 \geq \ldots \) where \( c_n = \frac{n}{n^2 + 2} \), and (ii) \( \lim_{n \to \infty} c_n = 0 \). To show (i), we need to show that \( c_n \geq c_{n+1} \) or \( \frac{n}{n^2 + 2} \geq \frac{n+1}{(n+1)^2 + 2} \). This inequality is equivalent to \( n^3 + 2n^2 + 3n = n((n + 1)^2 + 2) \geq (n + 1)(n^2 + 2) = n^3 + n^2 + 2n + 2 \) which holds for any \( n \geq 1 \). Thus (i) holds. As \( n \to \infty, \frac{n}{n^2 + 2} \to 0 \) so that (ii) holds. Now, the series in question converges by the Alternating Series Test.

(b) Determine whether the series converges absolutely or conditionally.

Consider the corresponding series of absolute values \( \sum_{n=1}^{\infty} \frac{n}{n^2 + 2} \). Since \( n^2 + 2 \leq 3n^2 \) for all \( n \geq 1 \), it follows that
\[ \sum_{n=1}^{\infty} \frac{n}{n^2 + 2} > \frac{1}{3} \sum_{n=1}^{\infty} \frac{1}{n}. \]
The latter series is one third of the Harmonic Series which diverges. It follows that the corresponding series of absolute values diverges or the original alternating series converges conditionally. One can also show that the corresponding series of absolute values diverges by showing that the improper integral \( \int_{1}^{\infty} \frac{x}{x^2 + 2} \, dx \) diverges.