Problem 1. (15 points) Let \( f \) be the function whose graph is shown.

(a) (5 points) Calculate \( \int_{1}^{6} f \), and show what it represents on the graph.

The relevant signed area is shaded on the graph at left.

The positive contribution is \( \int_{1}^{3} f = \frac{1}{2}(2)(4) = 4 \).

The negative contribution is \( \int_{3}^{6} f = -\frac{1}{2}(3)(1.5) = -2.25 \).

The total signed area is \( \int_{1}^{6} f = 4 - 2.25 = 1.75 \).

(b) (5 points) Calculate \( \int_{1}^{6} (1 - 2f) \). You may use your answer to (a).

Use the properties of integrals to split this up:

\[
\int_{1}^{6} (1 - 2f) = \int_{1}^{6} 1 - 2 \int_{1}^{6} f = \frac{5}{1.75}
\]

Here we computed \( \int_{1}^{6} 1 \) = the area under the horizontal line \( y = 1 \) between \( x = 1 \) and \( x = 6 \). This is a \( 5 \times 1 \) rectangle, whose area is 5.

(c) (5 points) Give an example of an interval \([a, b]\) on which the average value of \( f \) is zero, and explain how you can tell.

Examples are \([3, 4]\), \([4, 0]\), and \([5, 8]\), or any interval whose right endpoint is twice as far away from 3 as its left endpoint.

On these intervals, \( \int_{0}^{9} f = \int_{1}^{7} f = \int_{2}^{5} f = 0 \) because the positive contribution and negative contribution are areas of congruent triangles, cancelling each other out exactly. Since the integral evaluates to zero on these intervals, so does the average value.
Problem 2. (9 points) Finding the area underneath a semicircular curve is hard. The semicircle shown below is the graph of the function \( f(t) = \sqrt{1 - t^2} \). Its area function is
\[
A_f(x) = \frac{1}{2} \left( \arcsin x + x \sqrt{1 - x^2} \right).
\]

Use the area function to compute \( \int_{-1/2}^{1/2} \sqrt{1 - t^2} \). Be sure to show your work!

Remembering that \( A_f(x) = \int_0^x f \), we know that
\[
\int_{-1/2}^{1/2} f = \int_0^{1/2} f + \int_0^{-1/2} f = -\int_0^{-1/2} f + \int_0^{1/2} f = A_f(\frac{1}{2}) - A_f(-\frac{1}{2}),
\]
as indicated in the diagram above. We compute:
\[
A_f(\frac{1}{2}) = \frac{1}{2} \left( \arcsin \frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{1}{4}} \right) = \frac{1}{2} \left( \frac{\pi}{6} + \frac{\sqrt{3}}{4} \right)
\]
\[
- A_f(-\frac{1}{2}) = \frac{1}{2} \left( \arcsin \frac{-1}{2} - \frac{1}{2} \sqrt{1 - \frac{1}{4}} \right) = \frac{1}{2} \left( -\frac{\pi}{6} - \frac{\sqrt{3}}{4} \right)
\]
\[
A_f(\frac{1}{2}) - A_f(-\frac{1}{2}) = \frac{\pi}{6} + \frac{\sqrt{3}}{4} \approx 0.9566.
\]

Problem 3. (6 points) An excited honeybee quickly runs left and right along a honeycomb for 12.5 seconds. Its velocity \( v \) (in cm/sec) is shown on the graph below.

What is \( \int_0^{12.5} v \), how can you tell, and what does this number mean for the bee?

There is as much (positive) area above the horizontal axis as there is (negative) area beneath the horizontal axis. Therefore
\[
\int_0^{12.5} v = 0 \text{ cm},
\]
meaning that the bee winds up in the same place after 12.5 seconds as it started.