NAME:

Show ALL your work CAREFULLY.

(5pts) 1. A spherical balloon is filled with water. If the water is leaking out at 3 cc/min, find the rate at which the radius is changing when the volume is 1000cc. [Recall that the volume of a sphere of radius $R$ is $\frac{4}{3}\pi R^3$.]

The volume of a balloon of radius $R$ is $V = \frac{4}{3}\pi R^3$. By differentiating $V$ with respect to $t$, we have

$$\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3R^2 \cdot \frac{dR}{dt} = 4\pi R^2 \frac{dR}{dt}.$$  

Since the water is leaking out at 3cc/min, $\frac{dV}{dt} = -3$ cc/min. When $V = 1000$, $\frac{4}{3}\pi R^3 = 1000$. Solving for $R$, we obtain $R = 10 \left(\frac{3}{4\pi}\right)^{1/3}$. Thus, when $V = 1000$, we have

$$-3 = \frac{dV}{dt} = 4\pi (10) \left(\frac{3}{4\pi}\right)^{2/3} \frac{dR}{dt}.$$  

It follows that

$$\frac{dR}{dt} = -\frac{1}{100} \left(\frac{3}{4\pi}\right)^{1/3} \text{ cm/min}.$$  

(5pts) 2. Let $f(x)$ be a continuous function defined on the closed interval $[0, 1]$ so that $f(0) = -\frac{\pi}{2}$ and $f(1) = \frac{\pi}{2}$. Now consider the function $g(x) = \sin(f(x))$. Determine whether $g(x) = 0$ has a solution on $[0, 1]$, i.e., determine whether there is some number $c$ between 0 and 1 such that $g(c) = 0$. [Hint: use Intermediate Value Theorem.]

Since $f$ is continuous and the sine function is also continuous, it follows that the composite function $g(x) = \sin(f(x))$ is continuous. Now,

$$g(0) = \sin(f(0)) = \sin\left(-\frac{\pi}{2}\right) = -1 \quad \text{and} \quad g(1) = \sin(f(1)) = \sin\left(\frac{\pi}{2}\right) = 1.$$  

Applying the Intermediate Value Theorem to the continuous composite function $g$ on the interval $[0, 1]$, we know that there must exist some number $c$ between 0 and 1 such that $g(c) = 0$ since 0 is between $g(0) = -1$ and $g(1) = 1$. 

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