Read all of the following information before starting the exam:

- Show all work, clearly and in order if you want to get full credit (matrices can and should be reduced into RREF with calculator without showing steps unless otherwise indicated). I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point. I will take points off for rambling and for incorrect or irrelevant statements.
- This test has 7 problems and one Bonus and is worth 95 points. It is your responsibility to make sure that you have all of the pages!
- Good luck!
1. (20 points) Consider $A = \begin{bmatrix} 3 & 5 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 & 0 \\ 2 & -1 & -2 \\ -1 & 2 & 4 \end{bmatrix}$

a. (4 pts) Find the characteristic equation $c_A(x)$ for $A$. Find the eigenvalues of $A$.

b. (4 pts) Find the characteristic equation $c_B(x)$ for $B$. Find the eigenvalues of $B$.

c. (5 pts) Determine whether $A$ is diagonalizable. If so, find $P$ and $D$ such that $A = PDP^{-1}$.

d. (5 pts) Determine whether $B$ is diagonalizable. If so, find $P$ and $D$ such that $B = PDP^{-1}$.

e. (2 pts) What is $c_A(A)$ and what is $c_B(B)$? (You should NOT have to calculate this to answer this question).
2. (15 points) Given \( A = \begin{bmatrix}
0 & 1 & -2 & 2 & 0 \\
-1 & 3 & 0 & 1 & 6 \\
8 & -1 & 3 & 5 & 1
\end{bmatrix}\rightarrow \text{rref} \begin{bmatrix}
1 & 0 & 0 & 1 & -6 \\
0 & 1 & 0 & 0 & 2 \\
0 & 0 & 1 & -1 & 1
\end{bmatrix} \). Find a basis for the following.

a. (5 pts) \( \text{Col}(A) \)

b. (5 pts) \( \text{Nul}(A) \)

c. (5 pts) \( \text{Row}(A) \)

3. (9 points) Given an \( m \times n \) matrix \( A \). Prove (verify in generality) that \( \text{Nul}(A) \) is a subspace of \( \mathbb{R}^n \).

4. (10 points)

- If \( A \) is a \( 2 \times 2 \) matrix and has 2 distinct eigenvalues, then \( A \) must be diagonalizable.
- If matrices \( A \) and \( B \) have the same reduced row echelon form then \( \text{Row}(A) = \text{Row}(B) \).
- If \( H \) is a subspace of \( \mathbb{R}^3 \), then there is a \( 3 \times 3 \) matrix \( A \) such that \( H = \text{Col}(A) \).
- A subspace is also a vectorspace.
- A linearly independent set in a subspace \( H \) is a basis for \( H \).
5. (8 points) Consider vectors of the form \[
\begin{bmatrix}
a - 2b + 5c \\
2a + 5b - 8c \\
-a - 4b + 7c \\
3a + b + c
\end{bmatrix}.
\]

a. (3 pts) Show that the vectors form a subspace of \(\mathbb{R}^4\).

b. (5 pts) Find a basis for the set.
6. (15 points) Short answer.
   a. (5 pts) Prove that if 0 is an eigenvalue of $A$, then $A$ is not invertible (or an equivalent statement opposition to the IMT).

   b. (5 pts) Let $A$ be an $m \times n$ matrix, and let $B$ be an $n \times p$ matrix such that $AB = 0$. Show that \( \text{rank}(A) + \text{rank}(B) \leq n \) [Hint: one of the four subspaces $\text{Nul}(A)$, $\text{Col}(A)$, $\text{Nul}(B)$, $\text{Col}(B)$ is contained in one of the other three spaces.]

   c. (5 pts) Use coordinate vectors to test whether the set of polynomials span $\mathbb{P}_2$.
      \[
      \{1 - 3t + 5t^2, -3 + 5t - 7t^2, -4 + 5t - 6t^2, 1 - t^2\}
      \]
7. (18 points) Let \( \beta = \{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \} \) be a basis for \( M_{2\times 2}(\mathbb{R}) \).

a. (7 pts) Use coordinate vectors to prove whether \( \{ \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \} \) is a basis for \( M_{2\times 2}(\mathbb{R}) \).

Consider the transformation \( T : M_{2\times 2}(\mathbb{R}) \rightarrow M_{2\times 2}(\mathbb{R}) \) where \( T(A) = -A^T \). Recall an element in \( M_{2\times 2}(\mathbb{R}) \) can be represented as \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \).

b. (3 pts) What is the \( \text{Ker}(T) \)?

The transformation \( T \) has \( \lambda = 1 \) and \( \lambda = -1 \) as eigenvalues. (ie. There exists vectors \( \vec{x} \) such that \( T(\vec{x}) = \vec{x} \) or \( T(\vec{x}) = -\vec{x} \).

c. (4 pts) Without finding the matrix associated with \( T \) find the matrices in the eigenspace associated with \( \lambda = 1 \), then find a basis for the eigenspace.

d. (4 pts) Without finding the matrix associated with \( T \), find the matrices in the eigenspace associated with \( \lambda = -1 \), then find a basis for the eigenspace.
Bonus Question (2 Extra Credit Points):

The **left principal minors** of a matrix are the determinants of the matrices formed by deleting the last row and the last column of a matrix (creating a $3 \times 3$ matrix), then the last two columns and last two rows of the original matrix (creating a $2 \times 2$ matrix), then the last three rows and last three columns of the original matrix until the matrix is just $1 \times 1$ (a scalar).

A matrix is called positive definite if the determinant of the matrix and all the principal minors are positive.

Determine whether $A$ is positive definite.

$$A = \begin{bmatrix}
6 & 1 & 0 & -1 \\
2 & 2 & 0 & 1 \\
0 & -3 & 8 & 0 \\
0 & 1 & 0 & 5
\end{bmatrix}$$