**Problem 1.** (20 points) A minute hourglass contains 120 cm$^3$ of sand. When turned upside down, the sand empties from one chamber to the other at a constant rate, and in 60 seconds the top chamber has completely emptied.

This hourglass is shaped like a parabola; its height and volume are related by the equation shown on the figure at left. How rapidly is the height of the sand changing, 30 seconds after the hourglass begins to empty?

[Equation image: $V = \frac{1}{2} \pi h^2$]

The picture is already supplied for us; we just need to identify the important “players.”

**Step 1 — Identify the quantities involved.** There are only two in this problem: the height of sand in the hourglass $h$ (in centimeters), and the volume of sand in the hourglass $V$ (in cm$^3$).

**Step 2 — Know what you know, and know what you don't know.** Between $h$, $V$, $\frac{dh}{dt}$ and $\frac{dV}{dt}$, we are actually given none, and asked to compute $\frac{dh}{dt}$. However, we do have some information to help us figure out the others.

The sand is draining at a constant rate: in particular, all 120 cm$^3$ must drain in 60 seconds, so this rate is

$$\frac{dV}{dt} = \frac{-120 \text{ cm}^3}{60 \text{ sec}} = -2 \text{ cm}^3/\text{sec}.$$  

The negative sign is important! We use it because the volume of sand is decreasing.

Also, we can calculate that after 30 seconds, the volume of sand must have changed by

$$30 \times (-2) = -60 \text{ cm}^3.$$  

After 30 seconds, $V = 120 - 60 = 60 \text{ cm}^3$.

Furthermore, when $V = 60$, what is $h$? Using the equation in the figure,

$$60 = \frac{1}{2} \pi h^2$$

$$\frac{120}{\pi} = h^2$$

$$h = \sqrt{\frac{120}{\pi}}.$$  

Let’s organize our information in the following table.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$\sqrt{\frac{120}{\pi}}$ cm</th>
<th>$V$</th>
<th>60 cm$^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{dh}{dt}$</td>
<td>?</td>
<td>$\frac{dV}{dt}$</td>
<td>$-2 \text{ cm}^3/\text{sec}$</td>
</tr>
</tbody>
</table>

**Step 3 — Relate the quantities.** This is already done for us by the given formula relating height and volume:

$$V = \frac{1}{2} \pi h^2.$$  

**Step 4 — Implicit differentiation.** Differentiating the quantity equation with respect to $t$ gives us the rate equation:

$$\frac{dV}{dt} = \pi h \frac{dh}{dt}.$$  

Substituting the known quantities from our table permits us to solve for $\frac{dh}{dt}$:

$$-2 = \pi \sqrt{\frac{120}{\pi}} \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{-2}{\pi \sqrt{\frac{120}{\pi}}} = \frac{-2}{\sqrt{120\pi}} \approx -0.103 \text{ cm/sec}.$$  

So after 30 seconds, the height of sand in the hourglass is decreasing at a rate of $\frac{-2}{\sqrt{120\pi}} \approx -0.103$ centimeters per second.
Problem 2. (5 points) The equation \( e^x - 10x = 0 \) is impossible to solve algebraically.

(a) (3 points) Explain thoroughly why a solution must exist between \( x = 0 \) and \( x = 1 \).

Where is the function at \( x = 0 \) and \( x = 1 \)? If \( f(x) = e^x - 10x \), we observe that
\[
f(0) = e^0 - 10(0) = 1 \quad \text{and} \quad f(1) = e^1 - 10 \approx -7.2817.
\]

How does the function get from one point to another? Our function is made up of the difference of an exponential and a linear function. Both of those are continuous everywhere on the real line, so the result is a continuous function everywhere.

Does the intermediate value theorem apply? Yes — since we wish to guarantee that \( f(z) = 0 \) for some \( z \) between 0 and 1, we know \( f \) is continuous on this interval, and the desired target value (0) lies between \( f(0) = 1 \) and \( f(1) \approx -7.2817 \). Thus the intermediate value theorem guarantees that there is some \( z \) between 0 and 1 for which \( f(z) = 0 \).

(b) (2 points) Predict whether the solution lies between 0 and 0.5, or between 0.5 and 1.

Since we already know that \( f \) is continuous everywhere, we just need to know on which interval \( f \) changes sign. By the intermediate value theorem, \( f \) cannot change sign without passing through the value zero. Note that \( f(0.5) = e^{0.5} - 5 \approx -3.3513 \).

\[
\text{Solution must be here...}
\]

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>0.5</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>0</td>
<td>-3.3513</td>
<td>-7.2817</td>
</tr>
</tbody>
</table>

since \( f \) changes sign here.

The solution must lie between 0 and 0.5. (In fact, the actual solution is \( x \approx 0.11183 \).)

Problem 3. (5 points) Does the function \( g(x) = x^{0.4} \) have a stationary point between \( x = -1 \) and \( x = 1 \)?

Explain what, if anything, the mean value theorem says about the answer to this question. You may use the graph below in your explanation.

The mean value theorem relates the average rate of change of a function over an interval to its instantaneous rate of change somewhere inside the interval. Specifically, it guarantees that the two will match at some point, provided the function is continuous on the entire closed interval and differentiable everywhere on the interval except possibly the endpoints.

Here, the average rate of change of \( g(x) = x^{0.4} \) on the interval \([-1, 1]\) is
\[
\frac{g(1) - g(-1)}{1 - (-1)} = \frac{1^{0.4} - (-1)^{0.4}}{2} = 0.
\]

This is shown at left as the slope of the secant line joining those two points on the graph. If the mean value theorem applies, it will guarantee there is some point \( z \) between \(-1\) and \( 1 \) at which \( f'(z) = 0 \) — that is, a stationary point.

However, a look at the graph shows that no stationary point exists; there is no place on the graph for a horizontal tangent line. The mean value theorem does not apply, since \( g \) is not differentiable on the entire interval. In particular,
\[
g'(0) = 0.4(0)^{-0.6} = \frac{0.4}{0} \quad \text{does not exist.}
\]