1. A “True or False” question. In this problem, assume $A$ and $B$ are $4 \times 4$ square matrices. Suppose:
   i) $\text{rref}(A) = R$ where $R$ has exactly three leading-1’s,
   ii) $\text{rref}(B) = S$ and $S = I_4$. In the left margin, next to each statement, write “T” if the statement is always true and “F” if it’s not always true, for any such matrices $A$ and $B$.

a) $A$ is an invertible matrix.

b) The determinant of $B$ is non-zero.

c) The equation $Ax = b$ has infinitely many solutions $x$ for each $b \in \mathbb{R}^4$.

d) $\text{Nul}(A) = \text{Nul}(R)$.

e) $\text{rank}(A) = 3$.

f) $\text{Col}(B) = \text{Col}(S)$.

g) The number 0 is an eigenvalue for $A$.

h) The number 0 is an eigenvalue for $B$.

i) $\text{Nul}(B) = \{0\}$.

j) $A$ and $B$ are not row equivalent.

k) $A$ and $B$ are not similar.

l) In any basis of $\text{Nul}(A)$, there is exactly one vector.
2. Let \( B = \begin{bmatrix} 5 & 1 & 2 \\ -5 & 11 & 2 \\ 10 & -4 & 2 \end{bmatrix} \) Here are some facts about \( B \):

(i) \( v = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} \) is in \( \text{Nul}(B) \),

(ii) \( \lambda = 10 \) is an eigenvalue of \( B \),

(iii) \( u = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \) is an eigenvector for \( B \).

You should be able to answer these questions without finding any determinants and with maybe just one rref.

a) What is the eigenvalue corresponding to the eigenvector \( u \)?

b) What is the characteristic polynomial of \( B \) in factored form?

c) Show that \( B \) is diagonalizable by exhibiting \( P, D \) and \( P^{-1} \) that have the required properties.
3. Let \( C = \begin{bmatrix} 4 & 3 & -1 \\ 0 & 7 & -1 \\ 0 & 6 & 2 \end{bmatrix} \).

a) Find the characteristic polynomial of \( C \) in factored form starting from \( \det(C - \lambda I) \); show all your work.

b) One of the eigenvalues should have multiplicity 2. Find a basis for the eigenspace of that eigenvalue.

c) Without actually finding \( P, D \) and \( P^{-1} \), explain why \( C \) is diagonalizable.
4. Let $M$ be the $4 \times 4$ matrix here. FACT: The product of

$$
\begin{bmatrix}
6 & 1 & 2 & 7 \\
2 & 6 & 2 & -4 \\
2 & 3 & 6 & -3 \\
2 & -4 & 2 & 6
\end{bmatrix}
\begin{bmatrix}
12 & 7 & 5 & -3 \\
-1 & 2 & 6 & 2 \\
0 & 6 & 7 & 1 \\
7 & 0 & 0 & 2
\end{bmatrix}
\begin{bmatrix}
10 & 0 & 10 & -2 \\
0 & 0 & 12 & -1 \\
-1 & 0 & 14 & 6 \\
0 & 0 & 0 & 3
\end{bmatrix}
$$

is

$$
\begin{bmatrix}
120 & 56 & 50 & 0 \\
-10 & 38 & 60 & 0 \\
0 & 56 & 70 & 0 \\
70 & 18 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
4 & 0 & 100 & 20 \\
0 & 0 & 0 & -10 \\
-4 & 0 & 140 & 20 \\
0 & 0 & 0 & 30
\end{bmatrix}
$$

(a) Use this fact to find the eigenvalues of $M$, and bases for their respective eigenspaces. Note the fact contains useful and not useful information!

(b) Is $M$ diagonalizable? (Y/N)
5. Let \( D = \begin{bmatrix} a & 0 & 1 \\ c & 2 & 0 \\ 1 & 0 & e \end{bmatrix} \) and suppose \( \det(D) = -10 \). Find each of the following:

(a) \( \det(D^3) \)  
(b) \( \det(3D) \)

(c) \( \det(D+D) \)  
(d) \( \det(D^{-1}) \)

(e) \( \det(D) \)  
(f) \( \det(D^T) \)

(g) \( ae \)  
(h) \( \det \left( \begin{bmatrix} a & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & e \end{bmatrix} \right) \)