Math 106 BC

Quiz 06 page 1

11/12/2010 Name

Suggested solution

Circle your Section: B (11am) C (noon)

1. Consider the sequence \( \{a_k\}_{k=1}^{\infty} \) given by \( a_k = \frac{3^k - 1}{3^k} \).

1a. What are the first four terms of the sequence?

\[
\frac{2}{3}, \frac{8}{9}, \frac{26}{27}, \frac{80}{81}
\]

1b. What is the limit \( L \) of this sequence? \( \boxed{1} \)

1c. Find \( N \) such that \( |a_n - L| < \epsilon \) but if \( n > N \), then \( |a_n - L| < \epsilon \). This may just involve writing out enough terms of the sequence until you “see where it happens”.

\[
|a_6 - 1| = \left| \frac{3^6 - 1}{3^6} - 1 \right| = \left| \frac{728}{729} - 1 \right| = \left| \frac{-1}{729} \right| < 0.001
\]

but \( |a_7 - 1| = \left| \frac{3^7 - 1}{3^7} - 1 \right| = \left| \frac{2186}{2187} - 1 \right| = \left| \frac{-1}{2187} \right| < 0.001 \) (and all the terms after this one even closer to 1) \( \therefore N = 6 \)

1d. Bonus Repeat (1c) for \( \epsilon = 1/1,000,000,000,000,000 = 1/(one \ trillion) \). Show your computations!

(You need to solve \( \left| \frac{3^k - 1}{3^k} - 1 \right| < 10^{-12} \); use logs to find \( K > 25.15 \) \( \therefore N = 25 \))

2a. Using the notation developed in class, find \( a \) and \( r \) for the following geometric series:

\[
80 + 32 + 64/5 + 128/25 + 256/125 + \ldots
\]

If it is a geometric series, then \( r \) satisfies \( 80r = 32 \), \( 32r = \frac{64}{5} \), etc.

So \( r = \frac{32}{80} = \frac{\frac{64}{5}}{10.5} = \frac{8}{13} \); you can check that the terms follow this rule, \( a_{k+1} = \frac{2}{5} a_k \)

So \( a = 80 \), \( r = \frac{2}{5} \)

2b. What's the next term in the series?

\[
\frac{256 \times 2}{125} \times 5 = \frac{512}{625}
\]

2c. Does the series in (2a) converge? If so, to what?

Since \( |r| < 1 \), yes, it converges, to \( \frac{80}{1 - \frac{2}{5}} = \frac{5}{3} \times 80 = \frac{400}{3} = 133\frac{1}{3} \)

2d. If the "80" was not actually the first term of the series, what would the term in front of it be?

Call that new term \( T \). Since \( T \cdot \frac{2}{5} = 80 \), we have \( T = \frac{5}{2} \cdot 80 = 5 \cdot 40 = 200 \)

3a. Explicitly, what is the harmonic series? (give the first 5 terms).

\[
1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots
\]

3b. It's a fact that if a series \( \sum_{k=1}^{\infty} a_k \) converges, then the "individual terms" \( a_1, a_2, a_3, a_4 \ldots \) have to "go to zero". What does the harmonic series have to do with the converse of this fact?

The converse of the fact is: "If the sequence \( a_1, a_2, a_3, \ldots \) has a limit of 0, then the corresponding series \( a_1 + a_2 + a_3 + \ldots \) converges." But this statement is false, and the harmonic series is the classic counterexample: Even though the sequence \( \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \to 0 \), the series \( 1 + \frac{1}{2} + \frac{1}{3} + \ldots \) diverges.