Show ALL your work CAREFULLY.

Let

\[ A = \begin{bmatrix} 6 & -3 \\ 2 & 1 \end{bmatrix}. \]

(a) Find the eigenvalues of \( A \).

The eigenvalues of \( A \) are the solutions of the characteristic equation 
\( \det(A - \lambda I) = 0 \).

Since

\[ \det(A - \lambda I) = \det \begin{bmatrix} 6 - \lambda & -3 \\ 2 & 1 - \lambda \end{bmatrix} = \lambda^2 - 7\lambda + 12 = (\lambda - 4)(\lambda - 3), \]

it follows that the eigenvalues are 4 and 3.

(b) For each of the eigenvalue(s) found in (a), find the corresponding eigenspace.

For any eigenvalue \( \lambda \), the corresponding eigenspace is the null space of the matrix \( (A - \lambda I) \).

For \( \lambda = 4 \), consider

\[ A - 4I = \begin{bmatrix} 2 & -3 \\ 2 & -3 \end{bmatrix} \sim \begin{bmatrix} 2 & -3 \\ 0 & 0 \end{bmatrix}. \]

It follows that the null space of \( A - 4I \) is \( \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\} \).

Similarly for \( \lambda = 3 \), consider

\[ A - 3I = \begin{bmatrix} 3 & -3 \\ 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}. \]

It follows that the null space of \( A - 3I \) is \( \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \).

(c) Is \( A \) diagonalizable? If so, find an invertible matrix \( P \) such that \( P^{-1}AP \) is diagonal.

Since \( A \) has two distinct eigenvalues and thus two linearly independent eigenvectors, \( A \) is diagonalizable. The matrix \( P = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \) has the property that \( A = PDP^{-1} \) where

\[ D = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}. \]

It is straightforward to check that \( AP = PD \).