1. Find $dy/dx$ for each of the following.

(a) $y = x^2 + 2x + e^2 + e^{2x} + \ln 2 + \ln (2x) + \arctan 2$

$$\frac{dy}{dx} = 2x + (\ln 2)2x + 2e^{2x} + \frac{1}{2x}$$

Note that $e^2$, $\ln 2$, and $\arctan 2$ are constants.

(b) $y = \sqrt{x} \cdot \arctan (5x)$

$$\frac{dy}{dx} = \frac{1}{2}x^{-1/2} \arctan (5x) + \sqrt{x} \cdot \frac{1}{1 + (5x)^2} \cdot 5 = \frac{\arctan (5x) + 5\sqrt{x}}{2x^{1/2} + 1 + 25x^2}$$

(c) $y = \ln (\tan (2 \cos (x^2)))$

$$\frac{dy}{dx} = 1 \frac{\tan (2 \cos (x^2)) \cdot \sec^2 (2 \cos (x^2)) \cdot \ln 2 (2 \cos (x^2)) \cdot (-\sin (x^2)) \cdot 2x}{\tan (2 \cos (x^2))}$$

(d) $y = \frac{x + e^\pi}{\cos 4 + \sin^5 (6x)}$

Note that $e^\pi$ and $\cos 4$ are constants.

$$\frac{dy}{dx} = \frac{(1)(\cos 4 + \sin^5 (6x)) - (x + e^\pi)(5 \sin^4 (6x) \cdot \cos (6x) \cdot 6)}{(\cos 4 + \sin^5 (6x))^2}$$

Recall that $\sin^5 (6x) = (\sin (6x))^5$.

2. Consider the curve defined by $x^3 + y^3 = \frac{9}{2}xy$ (known as the Folium of Descartes).

(a) Find $dy/dx$. Use implicit differentiation.

$$3x^2 + 3y^2 \frac{dy}{dx} = \frac{9}{2}y + \frac{9}{2}x \frac{dy}{dx}$$

$$3y^2 \frac{dy}{dx} - \frac{9}{2} \frac{dy}{dx} = \frac{9}{2}y - 3x^2$$

$$\frac{dy}{dx} \left(3y^2 - \frac{9}{2} \frac{2}{x}\right) = \frac{9}{2}y - 3x^2$$

$$\frac{dy}{dx} = \frac{\frac{9}{2}y - 3x^2}{3y^2 - \frac{9}{2}x^2}$$

(b) Verify that the point (1,2) is on the curve above.

We must check to see if the values $x = 1$ and $y = 2$ satisfy the equation above.

$$x^3 + y^3 = \frac{9}{2}xy$$

$$1^3 + 2^3 = \frac{9}{2} \cdot 1 \cdot 2$$

$$9 = 9$$

Thus, the point (1,2) is on the curve.

(c) Find the equation of the tangent line at the point (1,2).

We want $y = mx + b$.

$$m = \frac{\frac{9}{2} \cdot \frac{2}{5} - 3 \cdot \frac{1}{2}}{3 \cdot \frac{2}{5} - \frac{1}{2}} = \frac{4}{5} \cdot \frac{1}{2} = \frac{4}{5} \cdot \frac{2}{3} = \frac{5}{3}$$

so $y = \frac{4}{5}x + b$.

Now plug in $x = 1$ and $y = 2$ to find $b$.

$$2 = \frac{4}{5} \cdot 1 + b \Rightarrow \frac{6}{5} = b$$

Therefore, we have $y = \frac{4}{5}x + \frac{6}{5}$. 

3. Evaluate the following limits.
Throughout this solution, the symbol ★ will stand for whatever notation your instructor prefers for using L’Hopital’s Rule on the indeterminate form 0/0; this may be \( \frac{0}{0} \) or \( \frac{\infty}{\infty} \) or \( \frac{\infty}{-\infty} \) or “has the form \( \frac{0}{0} \) and so, by L’Hopital’s Rule, is equal to” or something else. The symbol ▽ will serve the same purpose for the indeterminate forms \( \infty/\infty \) and \(-\infty/\infty\).

(a) \( \lim_{x \to 1} \frac{x^3 - 1}{7 - 7x} \) ★ \( \lim_{x \to 1} \frac{3x^2}{-7} = \frac{3}{-7} = -\frac{3}{7} \)

(b) \( \lim_{x \to 0} \frac{1 - \cos 2x}{3x} = 0 \) ★ \( \lim_{x \to 0} \frac{-2 \cdot x^2}{3x} = \frac{-2}{3} = 0 \)

(c) \( \lim_{x \to 0^+} x^2 \ln x \)
This is of the indeterminate form 0 \( \cdot (-\infty) \) so we rewrite the function as a fraction in order to use L’Hopital’s Rule.

\( \lim_{x \to 0^+} x^2 \ln x = \lim_{x \to 0^+} \frac{\ln x}{\frac{1}{x^2}} = \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{2}{x^3}} = \lim_{x \to 0^+} \frac{1}{x} \cdot \frac{-x^3}{-2} = \lim_{x \to 0^+} \frac{x^2}{2} = 0 \)

(d) \( \lim_{x \to 0} \frac{1 - \cos 4x}{5x^2} \) ★ \( \lim_{x \to 0} \frac{4 \sin 4x}{10x} = \frac{16 \cos 4x}{10} = 16 \cdot \frac{8}{10} = \frac{8}{5} \)

(e) \( \lim_{x \to \infty} \frac{x^2}{2x \cdot 2x} \) ★ \( \lim_{x \to \infty} \frac{2}{\ln 2 \cdot 2^x} \) ★ \( \lim_{x \to \infty} \frac{2}{\ln 2 \cdot \ln 2 \cdot 2^x} = 0 \)

4. Suppose that \( y = f(t) \) is a solution to the differential equation \( y' = \frac{1}{\pi} \arcsin t + y^2 \) and that \( f \left( \frac{\sqrt{2}}{2} \right) = \frac{1}{2} \). Find the equation of the tangent line to \( f \) at \( \left( \frac{\sqrt{2}}{2}, \frac{1}{2} \right) \).

We want \( y = mx + b \).

\( m = \frac{1}{\pi} \arcsin \frac{\sqrt{2}}{2} + \left( \frac{1}{2} \right)^2 = \frac{1}{\pi} \cdot \frac{\pi}{4} + \frac{1}{4} = \frac{1}{2} \), so \( y = \frac{1}{2}x + b \).

Now plug in \( x = \frac{\sqrt{2}}{2} \) and \( y = \frac{1}{2} \) to find \( b \).

\( \frac{1}{2} = \frac{1}{2} \cdot \frac{\sqrt{2}}{2} + b \Rightarrow \frac{1}{2} - \frac{\sqrt{2}}{4} = b \)

Therefore, we have \( y = \frac{1}{2}x + \frac{1}{2} - \frac{\sqrt{2}}{4} \).

5. Find the following.

(a) an antiderivative of \( y = \frac{5}{\sqrt{1 - 9x^2}} + x^3 + \cos(2x) + e^x \)
\( \frac{5 \arcsin 3x}{3} + \frac{x^4}{4} + \frac{\sin 2x}{2} + e^x + C \)

(b) \( \tan(\arccos x) \) (rewritten as an algebraic expression - no trigonometric functions)
Let \( \theta = \arccos x \). That is, \( \theta \) is the angle whose cosine is \( x \).

\[
\begin{align*}
\theta & = \arccos x \\
\sin \theta & = \sqrt{1 - x^2} \\
x^2 + y^2 & = 1^2 \\
y & = \sqrt{1 - x^2}
\end{align*}
\]
\[ \tan(\arccos x) = \tan \theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x} = \frac{\sqrt{1-x^2}}{x} \]

6. Consider the function \( f(x) = x^4e^x \) with domain all real numbers.

(a) Find the \( x \)-value(s) of all roots (\( x \)-intercepts) of \( f \).

The equation \( x^4e^x = 0 \) means \( x^4 = 0 \) (that is, \( x = 0 \)) or \( e^x = 0 \) (no solution), so the only root is at \( x = 0 \).

(b) Find the \( x \)- and \( y \)-value(s) of all critical points and identify each as a local max, local min, or neither.

\[
\begin{align*}
f' (x) &= 4x^3e^x + x^4e^x \\
0 &= x^3e^x(4 + x) \\
\Rightarrow x &= 0, -4 \\
\end{align*}
\]

Note that \( e^x \) is never 0.

\[
\begin{array}{c|c|c|c}
x & x < -4 & -4 < x < 0 & 4 < x \\
\hline
f' & \text{positive} & \text{negative} & \text{positive} \\
f & \\
\end{array}
\]

\( y \)-values: \( f(-4) = 256e^{-4} \approx 4.689, f(0) = 0 \)
So, \( f \) has a local maximum at \((-4,256e^{-4})\) and a local minimum at \((0,0)\).

(c) Find the \( x \)- and \( y \)-value(s) of all global extrema and identify each as a global max or global min.

There is a global minimum at \((0,0)\). There is no global maximum because as \( x \to \infty, f(x) \to \infty \).
Note that as \( x \to -\infty, f(x) \to 0 \). You can verify this by using L'Hopital's Rule on \( x^4/e^{-x} \).

(d) Find the \( x \)-value(s) of all inflection points.

\[
\begin{align*}
f'' (x) &= 12x^2e^x + 4x^3e^x + 4x^3e^x + x^4e^x \\
0 &= e^x(x^4 + 8x^3 + 12x) \\
0 &= e^x(x^2 + 8x + 12) \\
0 &= e^x(x^2 + 2)(x + 6) \\
\Rightarrow x &= 0, -2, -6 \\
\end{align*}
\]

\[
\begin{array}{c|c|c|c|c}
x & x < -6 & -6 < x < -2 & -2 < x < 0 & 0 < x \\
\hline
f'' & \text{positive} & \text{negative} & \text{positive} & \text{positive} \\
f & \text{concave up} & \text{concave down} & \text{concave up} & \text{concave up} \\
\end{array}
\]

So, the \( x \)-values of the inflection points of \( f \) are \( x = -2 \) and \( x = -6 \) but NOT \( x = 0 \).

(e) Sketch \( f \).
7. How would your answers to the previous question change if the domain of \( f \) were \([-10, 10]\)?
There would be a global maximum at \((10, 10^4 e^{10})\). (And the graph would be restricted to \(-10 \leq x \leq 10\)).

8. Use Newton’s Method with an initial guess of \( x_0 = -1 \) to find the next three approximations to a solution of \( x^3 + x + 1 = 0 \). Then test your final approximation to see if it appears to be close to a root.
Recall that \( x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 + x_n + 1}{3x_n^2 + 1} \).
Using a calculator with initial guess \( x_0 = -1 \), we get the following.
\[
\begin{align*}
x_0 &= -1 \\
x_1 &= -0.75 \\
x_2 &= -0.68604... \\
x_3 &= -0.68233...
\end{align*}
\]
Check: \((-0.68233)^3 + (-0.68233) + 1 \approx 0.0000526\), so our approximation seems to be a good one.

9. The rate of change of a population \( P(t) \) of eels is proportional to the size of the population. When the population is 40000, it is growing at a rate of 400 eels per year. At time \( t = 0 \), the population is 10000.
   (a) Write a differential equation whose solution is \( P(t) \).
   Rate of change \( (P') \) is (=) proportional to \( (k) \) size of population \( (P) \) means \( P' = kP \).
   What’s the value of \( k \)? When \( P = 40000 \), we know \( P' = 400 \). That is, \( 400 = k \cdot 40000 \), so \( k = .01 \).
   Thus, we have \( P' = .01P \).
   (b) Solve your differential equation.
   The general solution is \( P(t) = Ae^{.01t} \).
   What’s the value of \( A \)? When \( t = 0 \), we know \( P = 10000 \). That is, \( 10000 = Ae^0 = A \), so \( A = 10000 \).
   Thus, we have \( P(t) = 10000e^{.01t} \).
   (c) When will the population reach 60000?
   \[
   \begin{align*}
   60000 &= 10000e^{.01t} \\
   6 &= e^{.01t} \\
   \ln 6 &= \ln e^{.01t} \quad \text{Take ln of each side.} \\
   \ln 6 &= .01t \quad \text{Recall that } \ln e^z = z. \\
   t &= 100 \ln 6 \approx 179.176 \text{ years}
   \end{align*}
   \]

10. You are planning to build a box-shaped aquarium with no top and with two square ends. Your budget is $288. If the glass for the sides costs $12 per square foot and the opaque material for the bottom costs $3 per square foot, what dimensions will maximize the volume? Be sure to show how you know you have found the maximum.
**Goal:** Maximize volume

**Objective function:** \( V = x \cdot x \cdot y = x^2 y \)

We need to get this down to a function of just one variable, so we use the constraint equation:

\[
288 = 3xy + 12 \cdot 2x^2 + 12 \cdot 2xy
\]

\[
288 = 27xy + 24x^2
\]

\[
288 - 24x^2 = 27xy
\]

\[
\frac{288 - 24x^2}{27x} = y
\]

Substituting this back into the objective function gives

\[
V = x^2 y = x^2 \cdot \frac{288 - 24x^2}{27x} = x \cdot \frac{288 - 24x^2}{27} = \frac{1}{27} (288x - 24x^3).
\]

Now that we have \( V \) as a function of just one variable, we find its maximum.

\[
V'(x) = \frac{1}{27} (288 - 72x^2)
\]

\[
0 = \frac{1}{27} (288 - 72x^2)
\]

\[
0 = (288 - 72x^2)
\]

\[
72x^2 = 288
\]

\[
x^2 = \frac{288}{72} = 4
\]

\[
x = 2
\]

We discard \( x = -2 \) because lengths must be nonnegative.

Since \( V' \) is positive for \( x < 2 \) and negative for \( 2 < x \), we know that the maximum occurs at \( x = 2 \).

And \( y = \frac{288 - 24x^2}{27x} = \frac{288 - 24 \cdot 2^2}{27 \cdot 2} = \frac{32}{9} \), so the dimensions are 2 by 2 by \( \frac{32}{9} \).