Math 105: Review for Exam II - Solutions

1. Find $dy/dx$ for each of the following.
   (a) $y = x^2 + 2x + e^2 + e^{2x} + \ln 2 + \ln (2x) + \arctan 2$

   $$\frac{dy}{dx} = 2x + (\ln 2)2x + 2e^{2x} + \frac{1}{2x} \cdot 2$$

   Note that $e^2$, $\ln 2$, and $\arctan 2$ are constants.

   (b) $y = \sqrt{x} \cdot \arctan (5x)$

   $$\frac{dy}{dx} = \frac{1}{2}x^{-1/2} \arctan(5x) + \sqrt{x} \cdot \frac{1}{1 + (5x)^2} \cdot 5 = \frac{\arctan(5x)}{2x^{1/2}} + \frac{5\sqrt{x}}{1 + 25x^2}$$

   (c) $y = \ln(\tan(2\cos(2x^2)))$

   $$\frac{dy}{dx} = \frac{1}{\tan(2\cos(2x^2))} \cdot \sec^2(2\cos(2x^2)) \cdot \ln 2(2\cos(2x^2)) \cdot (-\sin(x^2)) \cdot 2x$$

   (d) $y = \frac{x + e^\pi}{\cos 4 + \sin^5(6x)}$

   Note that $e^\pi$ and $\cos 4$ are constants.

   $$\frac{dy}{dx} = \frac{(1)(\cos 4 + \sin^5(6x)) - (x + e^\pi)(5\sin(6x) \cdot \cos(6x) \cdot 6)}{(\cos 4 + \sin^5(6x))^2}$$

   Recall that $\sin^5(6x) = (\sin(6x))^5$.

2. Consider the curve defined by $x^3 + y^3 = \frac{9}{2}xy$ (known as the Folium of Descartes).

   (a) Find $dy/dx$. Use implicit differentiation.

   $$3x^2 + 3y^2 \frac{dy}{dx} = \frac{9}{2}y + \frac{9}{2}x \frac{dy}{dx}$$

   $$3y^2 \frac{dy}{dx} - \frac{9}{2} \frac{dy}{dx} = \frac{9}{2}y - 3x^2$$

   $$\frac{dy}{dx} \left(3y^2 - \frac{9}{2} \right) = \frac{9}{2}y - 3x^2$$

   $$\frac{dy}{dx} = \frac{\frac{9}{2}y - 3x^2}{3y^2 - \frac{9}{2} \cdot \frac{9}{2} \cdot \frac{9}{2}}$$

   (b) Verify that the point $(1,2)$ is on the curve above.

   We must check to see if the values $x = 1$ and $y = 2$ satisfy the equation above.

   $$x^3 + y^3 = \frac{9}{2}xy$$

   $$1^3 + 2^3 = \frac{9}{2} \cdot 1 \cdot 2$$

   $$9 = 9$$

   Thus, the point $(1,2)$ is on the curve.

   (c) Find the equation of the tangent line at the point $(1,2)$.

   We want $y = mx + b$.

   $$m = \frac{\frac{9}{2} \cdot 2 - 3 \cdot 1^2}{3 \cdot 2^2 - \frac{9}{2} \cdot \frac{9}{2}} = \frac{4}{5}$$

   So $y = \frac{4}{5}x + b$.

   Now plug in $x = 1$ and $y = 2$ to find $b$.

   $$2 = \frac{4}{5} \cdot 1 + b \Rightarrow \frac{6}{5} = b$$

   Therefore, we have $y = \frac{4}{5}x + \frac{6}{5}$. 

3. Evaluate the following limits.
Throughout this solution, the symbol ★ will stand for whatever notation your instructor prefers for using L’Hopital’s Rule on the indeterminate form 0/0; this may be “0/0” or $\frac{L'}{H'}$ or $\frac{H}{L}$ or = “0/0” or “has the form ‘0/0’ and so, by L’Hopital’s Rule, is equal to” or something else. The symbol ◊ will serve the same purpose for the indeterminate forms $\infty/\infty$ and $-\infty/\infty$.

(a) $\lim_{x\to-1} \frac{x^3 - 1}{7 - 7x} ★ \lim_{x\to-1} \frac{3x^2}{-7} = \frac{3}{-7} = -\frac{3}{7}$

(b) $\lim_{x\to0} \frac{1 - \cos 2x}{3x} = 0 \quad 0 = 0 \quad \text{Can’t use (and don’t need) L’Hopital’s Rule!}$

(c) $\lim_{x\to0} \frac{1 - \cos 4x}{5x^2} ★ \lim_{x\to0} \frac{4\sin 4x}{10x} ★ \lim_{x\to0} \frac{16 \cos 4x}{10} = \frac{16}{10} = \frac{8}{5}$

(d) $\lim_{x\to\infty} \frac{x^2}{2e} \left(\lim_{x\to\infty} \frac{2x}{\ln 2 \cdot 2^x} \right) \left(\lim_{x\to\infty} \frac{2}{\ln 2 \cdot 2^x} \right) = 0$

4. Find the following.

(a) an antiderivative of $y = \frac{5}{\sqrt{1 - 9x^2}} + x^3 + \cos(2x) + e^3$

$\frac{5 \arcsin 3x}{3} + \frac{x^4}{4} + \frac{\sin 2x}{2} + e^3x + C$

(b) $\tan(\arccos x)$ (rewritten as an algebraic expression - no trigonometric functions)

Let $\theta = \arccos x$. That is, $\theta$ is the angle whose cosine is $x$.

\[
\begin{array}{c}
\theta \\
\hline
x \quad y \\
\hline
1 \\
\end{array}
\]

$x^2 + y^2 = 1^2 \Rightarrow y = \sqrt{1 - x^2}$

$\tan(\arccos x) = \tan \theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x} = \frac{\sqrt{1 - x^2}}{x}$

5. Consider the function $f(x) = x^4e^x$ with domain all real numbers.

(a) Find the $x$-value(s) of all roots (x-intercepts) of $f$.

The equation $x^4e^x = 0$ means $x^4 = 0$ (that is, $x = 0$) or $e^x = 0$ (no solution), so the only root is at $x = 0$.

(b) Find the $x$- and $y$-value(s) of all critical points and identify each as a local max, local min, or neither.

$f'(x) = 4x^3e^x + x^4e^x$

$0 = x^3e^x(4 + x)$

$\Rightarrow x = 0, -4 \quad \text{Note that } e^x \text{ is never 0.}$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-4$ &lt; $x$ &lt; $0$</td>
<td>negative</td>
</tr>
<tr>
<td>$4$ &lt; $x$</td>
<td>positive</td>
</tr>
</tbody>
</table>

$y$-values: $f(-4) = 256e^{-4} \approx 4.689, f(0) = 0$

So, $f$ has a local maximum at $(-4, 256e^{-4})$ and a local minimum at $(0, 0)$. 

(c) Find the \(x\)- and \(y\)-value(s) of all global extrema and identify each as a global max or global min.

There is a global minimum at \((0,0)\). There is no global maximum because as \(x \to \infty\), \(f(x) \to \infty\). Note that as \(x \to -\infty\), \(f(x) \to 0\). You can verify this by using L’Hôpital’s Rule on \(x^4/e^{-x}\).

(d) Find the \(x\)-value(s) of all inflection points.

\[
f''(x) = 12x^2e^x + 4x^3e^x + 4x^3e^x + x^4e^x
\]

Use Product Rule on each product in \(f'(x)\) above.

\[
0 = e^x(x^4 + 8x^3 + 12x^2)
\]

\[
0 = e^x(x^2 + 8x + 12)
\]

\[
0 = e^x(x + 2)(x + 6)
\]

\[\Rightarrow x = 0, -2, -6\]

\[
\begin{array}{c|c|c|c|c}
\hline
f'' & x < -6 & -6 < x < -2 & -2 < x < 0 & 0 < x \\
\hline
f & positive & negative & positive & positive \\
\hline
f & concave up & concave down & concave up & concave up \\
\hline
\end{array}
\]

So, the \(x\)-values of the inflection points of \(f\) are \(x = -2\) and \(x = -6\) but NOT \(x = 0\).

(e) Sketch \(f\).

6. How would your answers to the previous question change if the domain of \(f\) were \([-10,10]\)?

There would be a global maximum at \((10, 10^4 e^{10})\). (And the graph would be restricted to \(-10 \leq x \leq 10\)).

7. You are planning to build a box-shaped aquarium with no top and with two square ends. Your budget is $288. If the glass for the sides costs $12 per square foot and the opaque material for the bottom costs $3 per square foot, what dimensions will maximize the volume? Be sure to show how you know you have found the maximum.

\[
\begin{align*}
\text{Goal: Maximize volume} \\
\text{Objective function: } V &= x \cdot x \cdot y = x^2y
\end{align*}
\]
We need to get this down to a function of just one variable, so we use the constraint equation:

\[
\text{total cost } = (\text{cost of base}) + (\text{cost of two square ends}) + (\text{cost of two other sides})
\]
\[
288 = 3xy + 12 \cdot 2x^2 + 12 \cdot 2xy
\]
\[
288 = 27xy + 24x^2
\]
\[
288 - 24x^2 = 27xy
\]
\[
\frac{288 - 24x^2}{27x} = y
\]

Substituting this back into the objective function gives

\[
V = x^2 y = x^2 \cdot \frac{288 - 24x^2}{27x} = x \cdot \frac{288 - 24x^2}{27} = \frac{1}{27} (288x - 24x^3).
\]

Now that we have \(V\) as a function of just one variable, we find its maximum.

\[
V'(x) = \frac{1}{27} (288 - 72x^2)
\]
\[
0 = \frac{1}{27} (288 - 72x^2)
\]
\[
0 = (288 - 72x^2)
\]
\[
72x^2 = 288
\]
\[
x^2 = \frac{288}{72}
\]
\[
x = 2
\]

We discard \(x = -2\) because lengths must be nonnegative.

Since \(V'\) is positive for \(x < 2\) and negative for \(2 < x\), we know that the maximum occurs at \(x = 2\).

And \(y = \frac{288 - 24x^2}{27x} = \frac{288 - 24 \cdot 2^2}{27 \cdot 2} = \frac{32}{9}\), so the dimensions are 2 by 2 by \(\frac{32}{9}\).