Math 105D - Exam 2 - November 10, 2006

Instructions: Show all of your work and circle your final answers. Calculators are allowed, but notes and books are not.

1. (15 points) You have 54 ft² of material with which to build a box with a square base. One side (not the top or the bottom) of the box is to remain open. What is the maximum possible volume of the box? (Be sure to justify how you know this is a maximum.)

\[ V = x \cdot x \cdot y = x^2 y \quad \text{(objective)} \]

Surface Area = \( 54 = x^2 + x^2 + xy + xy + xy = 2x^2 + 3xy \quad \text{(constraint)} \)

\[ 54 = 2x^2 + 3xy \rightarrow y = \frac{54 - 2x^2}{3x} = \frac{18 - 2x}{3} \]

\[ V = x^2 y = x^2 \left( \frac{18}{x} - \frac{2x}{3} \right) = 18x - \frac{2x^3}{3} \]

\[ V' = 18 - 2x^2 \]

\[ V' = 0 \rightarrow 18 - 2x^2 = 0 \]

\[ -2x^2 = -18 \]

\[ x^2 = 9 \]

\[ x = \pm 3. \quad \text{But} \quad x > 0. \]

\[ \therefore x = 3. \]

\[ + \rightarrow - \quad V' \quad V'(1) = 16 \]

\[ V'(4) = -14 \]

So \( V \) increases to \( x = 3 \), then decreases.

So there is a max at \( x = 3 \).

\[ y = \frac{18}{x} - \frac{2x}{3} = 4 \]

\[ \rightarrow V = x^2 y = 36 \text{ ft}^3 = \text{max volume.} \]
2. (10 points) The equation $x^3 + y^3 = 7 + x$ defines a graph. Using implicit differentiation, find $\frac{dy}{dx}$.

\[
\frac{d}{dx} \left( x^3 + y^3 \right) = \frac{d}{dx} (7 + x)
\]

\[
3x^2 + y^3 \frac{dy}{dx} + 3y^2 \frac{dy}{dx} = 1
\]

\[
(x^3 + 3y^2) \frac{dy}{dx} = 1 - 3x^2 y
\]

\[
\frac{dy}{dx} = \frac{1 - 3x^2 y}{x^3 + 3y^2}.
\]

3. (10 points) Find the minimum and maximum values of the function $f(x) = \frac{x}{1 + x^2}$ on the interval $[-3, 2]$.

\[
f'(x) = \frac{(1 + x^2) - x(2x)}{(1 + x^2)^2} = \frac{1 + x^2 - 2x^2}{(1 + x^2)^2} = \frac{1 - x^2}{(1 + x^2)^2}.
\]

$f'(x) = 0$ when $1 - x^2 = 0 \quad \Rightarrow \quad x = \pm 1$.

$f'(x)$ does not change $(1 + x^2)^2 = 0$. Never happens.

Critical points: $x = 1, x = -1$. End points: $x = -3, x = 2$.

\[
f(1) = \frac{1}{2}, \quad f(-1) = -\frac{1}{2}, \quad f(-3) = -\frac{3}{10}, \quad f(2) = \frac{2}{5}.
\]

Global max at $x = 1$.
Global min at $x = -1$. 
4. (15 points) Suppose \( g(x) \) is a function where \( g(1) = 2, g'(1) = -4, g''(1) = 3 \).

(a) Find the equation of the tangent line to the graph of \( y = g(x) \) at \( x = 1 \).

\[
\begin{align*}
\text{Slope } & = g'(1) = -4, \\
\text{Point } & = (1, 2), \\
y - y_0 & = m(x - x_0) \\
y - 2 & = -4(x - 1) \\
y - 2 & = -4x + 4 \\
\overbrace{y} & = -4x + 6
\end{align*}
\]

(b) Suppose that \( f(x) = e^x g(x) \). Determine the concavity of \( f(x) \) at \( x = 1 \).

\[
\begin{align*}
\frac{d}{dx} [e^x g(x)] & = e^x g(x) + e^x g'(x) \quad \text{(product rule)} \\
\frac{d}{dx} [e^x g'(x)] & = e^x g'(x) + e^x g''(x) \quad \text{(again)} \\
f'(1) & = e \cdot 2 + e \cdot (-4) + e \cdot 3 \\
& = -3e < 0 \\
\fbox{So \( f(x) \) is concave down at \( x = 1 \).}
\end{align*}
\]

5. (8 points) Find all antiderivatives of \( h(x) = x^2 + \sqrt{x} + \frac{1}{1 + 4x^2} \).

\[
\begin{align*}
H(x) & = \frac{1}{3} x^3 + \frac{2}{3} x^{3/2} + \frac{1}{2} \arctan(2x) + C
\end{align*}
\]
6. (16 points)

(a) Calculate \( \lim_{{x \to \infty}} \frac{e^{2x} + x}{3x^2} = \frac{\infty}{\infty} \)

\[
\begin{align*}
\text{L'Hôpital's Rule:} \\
\lim_{{x \to \infty}} \frac{e^{2x} + 1}{6x} &= \frac{\infty}{\infty} \\
\lim_{{x \to \infty}} \frac{4e^{2x}}{6} &= \infty
\end{align*}
\]

(b) Calculate \( \lim_{{x \to \pi}} \frac{\sin(3x)}{x - \pi} = \frac{0}{0} \)

\[
\begin{align*}
\text{L'Hôpital's Rule:} \\
\lim_{{x \to \pi}} \frac{3 \cos(3x)}{1} &= \frac{3 \cos(3\pi)}{1} = -3
\end{align*}
\]
7. (10 points) For some function \( g(x) \), suppose we know that there is a root at \( x = a \) for some number \( a \) in the interval \((1, 4)\). Also suppose that \( g(x) \) is increasing and concave down on the interval \([1, 4]\).

(a) Draw the graph (labeled appropriately) of a function \( g(x) \) that satisfies the above properties.

(b) Using Newton's method, if our initial guess \( x_0 \) is in \([1, 4]\) and greater than \( a \), will \( x_1 \) be less than \( a \) or greater than \( a \), or can we not be sure? Explain. (Your graph above may be helpful.)

Since the graph is concave down, a guess to the right of \( x = a \) will have a tangent line that hits the \( x \)-axis to the left of the root \( x = a \).

Hence, \( x_1 \) will be less than \( a \).

(c) What if our initial guess \( x_0 \) is in \([1,4]\) and less than \( a \)? Explain.

The graph slopes are decreasing, so our tangent line at \( x_0 \) will be steeper than the graph, so the tangent line will hit the \( x \)-axis to the left of \( a \).

Hence, \( x_1 \) will be less than \( a \).
8. (16 points) Find derivatives of the following. DO NOT SIMPLIFY.

(a) If \( f(x) = (\sin(\sqrt{x}) + \ln x + 5^x)^{10} \), find \( f'(x) \).

\[
\begin{align*}
    f'(x) &= 10 \left( \sin(\sqrt{x}) + \ln x + 5^x \right)^9 \left( \cos(\sqrt{x}) \cdot \frac{1}{2} x^{-\frac{1}{2}} + \frac{1}{x} \right) \\
    \text{(chain rule)}
\end{align*}
\]

(b) If \( g(x) = \ln(\arctan(1/x)) \), find \( g'(x) \).

\[
\begin{align*}
g'(x) &= \frac{1}{\arctan(1/x)} \cdot \frac{d}{dx} \left( \arctan(1/x) \right) \\
    &= \frac{1}{\arctan(1/x)} \cdot \frac{1}{1 + (1/x)^2} \cdot \frac{d}{dx} \left( 1/x \right) \\
    &= \frac{1}{\arctan(1/x)} \cdot \frac{1}{1 + (1/x)^2} \cdot \left( -x^{-2} \right) \\
    \text{(chain rule)}
\end{align*}
\]

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